Systematic Circuit Analysis (T&R Chap 3)

Node-voltage analysis
Using the voltages of the each node relative to a ground node, write down a set of consistent linear equations for these voltages
Solve this set of equations using, say, Cramer’s Rule

Mesh current analysis
Using the loop currents in the circuit, write down a set of consistent linear equations for these variables. Solve.

This introduces us to procedures for systematically describing circuit variables and solving for them
Nodal Analysis

Node voltages
Pick one node as the ground node
Label all other nodes and assign voltages $v_A, v_B, \ldots, v_N$
and currents with each branch $i_1, \ldots, i_M$
Recognize that the voltage across a branch is the difference between the end node voltages
Thus $v_3 = v_B - v_C$ with the direction as indicated
Write down the KCL relations at each node
Write down the branch $i$-$v$ relations to express branch currents in terms of node voltages
Accommodate current sources
Obtain a set of linear equations for the node voltages
Nodal Analysis – Example 3-1 p.72

Apply KCL

Node A: \( i_0 - i_1 - i_2 = 0 \)
Node B: \( i_1 - i_3 + i_5 = 0 \)
Node C: \( i_2 - i_4 - i_5 = 0 \)

Write the element/branch eqns

\[
\begin{align*}
    i_0 &= i_{S0} \\
    i_1 &= G_1(v_A - v_B) \\
    i_2 &= G_2(v_A - v_C) \\
    i_3 &= G_3v_B \\
    i_4 &= G_4v_C \\
    i_5 &= i_{S2}
\end{align*}
\]

Substitute to get node voltage equations

Node A: 
\[
(G_1 + G_2)v_A - G_1v_B - G_2v_C = i_{S1}
\]

Node B: 
\[
-G_1v_A + (G_1 + G_3)v_B = i_{S2}
\]

Node C: 
\[
-G_2v_A + (G_2 + G_4)v_C = -i_{S2}
\]

Solve for \( v_A, v_B, v_C \) then \( i_0, i_1, i_2, i_3, i_4, i_5 \)
Systematic Nodal Analysis

\[
\begin{pmatrix}
G_1 + G_2 & -G_1 & -G_2 \\
-G_1 & G_1 + G_3 & 0 \\
-G_2 & 0 & G_2 + G_4
\end{pmatrix}
\begin{pmatrix}
v_A \\
v_B \\
v_C
\end{pmatrix}
= \begin{pmatrix}
 i_{S1} \\
 i_{S2} \\
-i_{S2}
\end{pmatrix}
\]

Writing node equations by inspection

Note that the matrix equation looks just like \( Gv = i \) for matrix \( G \) and vector \( v \) and \( i \)

\( G \) is symmetric (and non-negative definite)

Diagonal \((i,i)\) elements: sum of all conductances connected to node \( i \)

Off-diagonal \((i,j)\) elements: -conductance between nodes \( i \) and \( j \)

Right-hand side: current sources entering node \( i \)

There is no equation for the ground node – the column sums give the conductance to ground
Nodal Analysis Example 3-2 p.74

Node A:
Conductances
\[ \frac{G}{2B} + 2G_C = 2.5G \]
Source currents entering = \( i_S \)

Node B:
Conductances
\[ \frac{G}{2A} + \frac{G}{2C} + 2G_{ground} = 3G \]
Source currents entering = 0

Node C:
Conductances
\[ 2G_A + \frac{G}{2B} + G_{ground} = 3.5G \]
Source currents entering = 0

\[
\begin{pmatrix}
2.5G & -0.5G & -2G \\
-0.5G & 3G & -0.5G \\
-2G & -0.5G & 3.5G
\end{pmatrix}
\begin{pmatrix}
v_A \\
v_B \\
v_C
\end{pmatrix}
= \begin{pmatrix} i_S \\ 0 \\ 0 \end{pmatrix}
\]
Nodal Analysis – some points to watch

1. The formulation given is based on KCL with the sum of currents leaving the node

\[ 0 = i_{\text{total}} = G_{\text{AtoB}}(v_A - v_B) + G_{\text{AtoC}}(v_A - v_C) + \ldots + G_{\text{AtoGround}}v_A + i_{\text{leavingA}} \]

This yields

\[ 0 = (G_{\text{AtoB}} + \ldots + G_{\text{AtoGround}})v_A - G_{\text{AtoB}}v_B - G_{\text{AtoC}}v_C \ldots - i_{\text{enteringA}} \]
\[ (G_{\text{AtoB}} + \ldots + G_{\text{AtoGround}})v_A - G_{\text{AtoB}}v_B - G_{\text{AtoC}}v_C \ldots = i_{\text{enteringA}} \]

2. If in doubt about the sign of the current source, go back to this basic KCL formulation

3. This formulation works for independent current source

For dependent current sources (introduced later) use your wits
Solve this using standard linear equation solvers

Cramer’s rule
Gaussian elimination
Matlab
Nodal Analysis with Voltage Sources

Current through voltage source is not computable from voltage across it. We need some tricks!
They actually help us simplify things
Method 1 – source transformation

Then use standard nodal analysis – one less node!
Nodal Analysis with Voltage Sources 2

Method 2 – grounding one node

This removes the $v_B$ variable – simpler analysis
But can be done once per circuit
Nodal Analysis with Voltage Sources 3

Method 3

Create a *supernode*

Act as if A and B were one node

KCL still works for this node

Sum of currents entering supernode box is 0

Write KCL at all N-3 other nodes

N-2 nodes less Ground node

using \( v_A \) and \( v_B \) as usual

Write one supernode KCL

Add the constraint \( v_A - v_B = v_S \)

These three methods allow us to deal with all cases
Nodal Analysis Example 3-4 p. 76

This is method 1 – transform the voltage sources

Applicable since voltage sources appear in series with Rs

Now use nodal analysis with one node, A

\[
(G_1 + G_2 + G_3)v_A = G_1v_{S1} + G_2v_{S2}
\]

\[
v_A = \frac{G_1v_{S1} + G_2v_{S2}}{G_1 + G_2 + G_3}
\]
Rewrite in terms of $v_S$, $v_B$, $v_C$

This is method 2

\[ v_B = \frac{2.75v_S}{10.25}, \quad v_C = \frac{6.25v_S}{10.25} \]

Solve

\[ i_{in} = \frac{v_S - v_B}{2R} + \frac{v_S - v_C}{R/2} = \frac{11.75v_S}{10.25R} \]

\[ R_{in} = \frac{10.25R}{11.75} = 0.872R \]

What is the circuit input resistance viewed through $v_S$?

\[ v_A = v_S \]

\[-0.5Gv_A + 3Gv_B - 0.5Gv_C = 0\]

\[-2Gv_A - 0.5Gv_B + 3.5v_C = 0\]

\[ 3Gv_B - 0.5Gv_C = 0.5Gv_S \]

\[-0.5Gv_B + 3.5Gv_C = 2Gv_S \]
Method 3 – supernodes

KCL for supernode: \( i_1 + i_2 + i_3 + i_4 = 0 \)

Or, using element equations

\[
G_1 v_A + G_2 (v_A - v_B) + G_3 (v_C - v_B) + G_4 v_C = 0
\]

Now use \( v_B = v_{S2} \)

\[
(G_1 + G_3) v_A + (G_3 + G_4) v_C = (G_2 + G_3) v_{S2}
\]

Other constituent relation

\[
v_A - v_C = v_{S1}
\]

Two equations in two unknowns
Summary of Nodal Analysis

1. Simplify the cct by combining elements in series or parallel

2. Select as reference node the one with most voltage sources connected

3. Label node voltages and supernode voltages – do not label the nodes directly connected to the reference

4. Use KCL to write node equations. Express element currents in terms of node voltages and ICSs

5. Write expressions relating node voltages and IVSs

6. Substitute from Step 5 into equations from Step 4
   Write the equations in standard form

7. Solve using Cramer, gaussian elimination or matlab
Solving sets of linear equations

\[
\begin{pmatrix}
5 & -2 & -3 \\
-5 & 7 & -2 \\
-3 & -3 & 8
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
4 \\
-10 \\
6
\end{pmatrix}
\]


\[
\Delta = \begin{vmatrix}
5 & -2 & -3 \\
-5 & 7 & -2 \\
-3 & -3 & 8
\end{vmatrix} = 5 \begin{vmatrix}
7 & -2 \\
-3 & 8
\end{vmatrix} - (-5) \begin{vmatrix}
-2 & -3 \\
-3 & 8
\end{vmatrix} + (-3) \begin{vmatrix}
-2 & -3 \\
7 & -2
\end{vmatrix}
\]

\[= 5(7 \times 8 - (-3) \times (-2)) + 5((-2) \times 8 - (-3) \times (-3)) - 3((-2) \times (-2) - 7 \times (-3)) \]

\[= 250 - 125 - 75 = 50\]

\[
\Delta_1 = \begin{vmatrix}
4 & -2 & -3 \\
-10 & 7 & -2 \\
6 & -3 & 8
\end{vmatrix} = 4 \begin{vmatrix}
7 & -2 \\
-3 & 8
\end{vmatrix} - (-10) \begin{vmatrix}
-2 & -3 \\
-3 & 8
\end{vmatrix} + 6 \begin{vmatrix}
-2 & -3 \\
7 & -2
\end{vmatrix}
\]

\[= 4(7 \times 8 - (-3) \times (-2)) + 10((-2) \times 8 - (-3) \times (-3)) + 6((-2) \times (-2) - 7 \times (-3)) \]

\[= 200 - 250 + 150 = 100\]

x_1 = \frac{\Delta_1}{\Delta} = \frac{100}{50} = 2
Solving sets of linear equations (contd)

\[ \Delta_2 = \begin{vmatrix} 5 & 4 & -3 \\ -5 & -10 & -2 \\ -3 & 6 & 8 \end{vmatrix} = 5 \begin{vmatrix} -10 & -2 \end{vmatrix} - (-5) \begin{vmatrix} 4 & -3 \end{vmatrix} + (-3) \begin{vmatrix} 4 & -3 \end{vmatrix} \\
= -340 + 250 + 114 = 24 \]

\[ x_2 = \frac{\Delta_2}{\Delta} = \frac{24}{50} = 0.48 \]

\[ \Delta_3 = \begin{vmatrix} 5 & -2 & 4 \\ -5 & 7 & -10 \\ -3 & -3 & 6 \end{vmatrix} = 5 \begin{vmatrix} 7 & -10 \end{vmatrix} - (-5) \begin{vmatrix} -2 & 4 \end{vmatrix} + (-3) \begin{vmatrix} -2 & 4 \end{vmatrix} \\
= 60 - 0 + 24 = 84 \]

\[ x_3 = \frac{\Delta_3}{\Delta} = \frac{84}{50} = 1.68 \]

Notes:

This Cramer is not as much fun as Cosmo Kramer in *Seinfeld*
I do not know of any tricks for symmetric matrices
Solving Linear Equations – gaussian elimination

\[
\begin{pmatrix}
5 & -2 & -3 \\
-5 & 7 & -2 \\
-3 & -3 & 8
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
4 \\
-10 \\
6
\end{pmatrix}
\]

Augment matrix

\[
\begin{pmatrix}
5 & -2 & -3 & 4 \\
-5 & 7 & -2 & -10 \\
-3 & -3 & 8 & 6
\end{pmatrix}
\]

Row operations only

row_2 + row_1
row_3 + row_1 \times 3/5

\[
\begin{pmatrix}
5 & -2 & -3 & 4 \\
0 & 5 & -5 & -6 \\
0 & -21 & 31 & 42
\end{pmatrix}
\]

row_3 \times 5

\[
\begin{pmatrix}
5 & -2 & -3 & 4 \\
0 & 5 & -5 & -6 \\
0 & -21 & 31 & 42
\end{pmatrix}
\]

(row_2 + row_3 \times 5) \div 5

(row_1 + row_2 \times 2 + row_3 \times 3) \div 5
Solving Linear Equations - matlab

A=[5 -2 -3; -5 7 -2; -3 -3 8]

A =
5 -2 -3
-5 7 -2
-3 -3 8

B=[4;-10;6]

B =
4
-10
6

inv(A)

ans =
1.0000  0.5000  0.5000
0.9200  0.6200  0.5000
0.7200  0.4200  0.5000

inv(A)*B

ans =
2.0000
0.4800
1.6800

A\B

ans =
2.0000
0.4800
1.6800
Mesh Current Analysis

Dual of Nodal Voltage Analysis with KCL

Mesh Current Analysis with KVL

\[ \text{Mesh} = \text{loop enclosing no elements} \]

Restricted to Planar Ccts – no crossovers (unless you are really clever)

**Key Idea:** \( \text{If element } K \text{ is contained in both mesh } i \text{ and mesh } j \text{ then its current is } i_k = i_i - i_j \) where we have taken the reference directions as appropriate

Same old tricks you already know

\[ \begin{align*}
R_1 & \quad R_2 & \quad R_3 \\
+ & \quad + & \quad + \\
\text{v}_0 & \quad \text{v}_{S1} & \quad \text{v}_{S2} \\
\text{v}_1 & \quad \text{v}_2 & \quad \text{v}_3 \quad \text{v}_4 \\
\text{v}_0 - \text{v}_1 - \text{v}_3 & = 0 \\
\text{v}_3 - \text{v}_2 - \text{v}_4 & = 0 \\
(R_1 + R_3)i_A - R_3i_B & = \text{v}_{S1} \\
-R_3i_A + (R_2 + R_3)i_B & = -\text{v}_{S2} \\
\end{align*} \]

\[
\begin{pmatrix}
R_1 + R_3 & -R_3 \\
-R_3 & R_2 + R_3
\end{pmatrix}
\begin{pmatrix}
i_A \\
i_B
\end{pmatrix}
=
\begin{pmatrix}
\text{v}_{S1} \\
-\text{v}_{S2}
\end{pmatrix}
\]
Mesh Analysis by inspection \( Ri = v_S \)

Matrix of Resistances \( R \)
- Diagonal \( ii \) elements: sum of resistances around loop
- Off-diagonal \( ij \) elements: - resistance shared by loops \( i \) and \( j \)

Vector of currents \( i \)
- As defined by you on your mesh diagram

Voltage source vector \( v_S \)
- Sum of voltage sources \textit{assisting} the current in your mesh
  - If this is hard to fathom, go back to the basic KVL to sort these directions out

\[
\begin{bmatrix}
     R_1 + R_2 & 0 & -R_2 \\
     0 & R_3 + R_4 & -R_3 \\
    -R_2 & -R_3 & R_2 + R_3
\end{bmatrix}
\begin{bmatrix}
    i_A \\
    i_B \\
    i_C
\end{bmatrix}
= \begin{bmatrix}
    -v_{S2} \\
    v_{S2} \\
    -v_{S1}
\end{bmatrix}
\]
Mesh Equations with Current Sources

Duals of tricks for nodal analysis with voltage sources

1. Source transformation to equivalent

T&R Example 3-8 p. 91

\[
\begin{pmatrix}
6000 & -2000 \\
-2000 & 11000
\end{pmatrix}
\begin{pmatrix}
i_A \\
i_B
\end{pmatrix} =
\begin{pmatrix}
5 \\
-8
\end{pmatrix}
\]

\(i_A = 0.6290\) mA
\n\(i_B = -0.6129\) mA
\n\(i_O = i_A - i_B = 1.2419\) mA
Mesh Analysis with ICSs – method 2

Current source belongs to a single mesh

\[ 6000i_A - 2000i_B = 5 \]
\[ -2000i_A + 11000i_B - 4000i_C = 0 \]
\[ i_C = -2 \text{ mA} \]

Same equations!
Same solution
Mesh Analysis with ICSs – Method 3

Supermeshes – easier than supernodes
Current source in more than one mesh and/or not in parallel with a resistance

1. Create a supermesh by eliminating the whole branch involved
2. Resolve the individual currents last

\[ R_1 (i_B - i_A) + R_2 i_B + R_4 i_C + R_3 (i_C - i_A) = 0 \]
\[ i_A = i_S1 \]
\[ i_B - i_C = i_S2 \]
Summary of Mesh Analysis

1. Check if cct is planar or transformable to planar

2. Identify meshes, mesh currents & supermeshes

3. Simplify the cct where possible by combining elements in series or parallel

4. Write KVL for each mesh

5. Include expressions for ICSs

6. Solve for the mesh currents
Linearity & Superposition

Linear cct – modeled by linear elements and independent sources

Linear functions

Homogeneity: \( f(Kx) = Kf(x) \)
Additivity: \( f(x+y) = f(x) + f(y) \)

Superposition – follows from linearity/additivity

Linear cct response to multiple sources is the sum of the responses to each source

1. “Turn off” all independent sources except one and compute cct variables
2. Repeat for each independent source in turn
3. Total value of all cct variables is the sum of the values from all the individual sources
Superposition

Turning off sources

Voltage source

Turned off when $v=0$ for all $i$
a short circuit

Current source

Turned off when $i=0$ for all $v$
an open circuit

We have already used this in Thévenin and Norton equiv
Where are we now?

Finished resistive ccts with ICS and IVS

Two analysis techniques – nodal voltage and mesh current

Preference depends on simplicity of the case at hand

The aim has been to develop generalizable techniques for access to analytical tools like matlab

Where to now?

Active ccts with resistive elements – transistors, op-amps

Life starts to get interesting – design introduced

Capacitance and inductance – dynamic ccts

Frequency response – $s$-domain analysis

Filters