Laplace Transforms – recap for ccts

What’s the big idea?

1. Look at initial condition responses of ccts due to capacitor voltages and inductor currents at time $t=0$
   
   Mesh or nodal analysis with $s$-domain impedances (resistances) or admittances (conductances)
   
   Solution of ODEs driven by their initial conditions
   
   Done in the $s$-domain using Laplace Transforms

2. Look at forced response of ccts due to input ICSs and IVSs as functions of time
   
   Input and output signals $I_O(s) = Y(s)V_S(s)$ or $V_O(s) = Z(s)I_S(s)$
   
   The cct is a system which converts input signal to output signal

3. Linearity says we add up parts 1 and 2
   
   The same as with ODEs
Laplace transforms

Time domain ($t$ domain)

Linear cct

Differential equation

Classical techniques

Response signal

Complex frequency domain ($s$ domain)

Laplace transform $\mathcal{L}$

Inverse Laplace transform $\mathcal{L}^{-1}$

Algebraic equation

Algebraic techniques

Response transform

The diagram commutes

Same answer whichever way you go
Laplace Transform - definition

Function $f(t)$ of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt$$

$0^-$ limit is used to capture transients and discontinuities at $t=0$

$s$ is a complex variable $(\sigma + j\omega)$

There is a need to worry about regions of convergence of the integral

Units of $s$ are $\text{sec}^{-1} = \text{Hz}$

A frequency

If $f(t)$ is volts (amps) then $F(s)$ is volt-seconds (amp-seconds)
Laplace transform examples

Step function – unit Heavyside Function
After Oliver Heavyside (1850-1925)

\[ u(t) = \begin{cases} 
0, & \text{for } t < 0 \\
1, & \text{for } t \geq 0 
\end{cases} \]

\[
F(s) = \int_{0^-}^{\infty} u(t)e^{-st} \, dt = \int_{0^-}^{\infty} e^{-st} \, dt = -\left. \frac{e^{-st}}{s} \right|_0^\infty = -\left. \frac{e^{-(\sigma+j\omega)t}}{\sigma + j\omega} \right|_0^\infty = \frac{1}{s} \text{ if } \sigma > 0
\]

Exponential function
After Oliver Exponential (1176 BC- 1066 BC)

\[
F(s) = \int_0^{\infty} e^{-\alpha t} e^{-st} \, dt = \int_0^{\infty} e^{-(\sigma+\alpha)t} \, dt = -\left. \frac{e^{-(\sigma+\alpha)t}}{s + \alpha} \right|_0^\infty = \frac{1}{s + \alpha} \text{ if } \sigma > \alpha
\]

Delta (impulse) function \( \delta(t) \)

\[
F(s) = \int_{0^-}^{\infty} \delta(t)e^{-st} \, dt = 1 \text{ for all } s
\]
# Laplace Transform Pair Tables

<table>
<thead>
<tr>
<th>Signal</th>
<th>Waveform</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>impulse</td>
<td>$\delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>step</td>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>ramp</td>
<td>$tu(t)$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>exponential</td>
<td>$e^{-\alpha t}u(t)$</td>
<td>$\frac{1}{s+\alpha}$</td>
</tr>
<tr>
<td>damped ramp</td>
<td>$te^{-\alpha t}u(t)$</td>
<td>$\frac{1}{(s+\alpha)^2}$</td>
</tr>
<tr>
<td>sine</td>
<td>$\sin(\beta t)u(t)$</td>
<td>$\frac{\beta}{s^2+\beta^2}$</td>
</tr>
<tr>
<td>cosine</td>
<td>$\cos(\beta t)u(t)$</td>
<td>$\frac{s}{s^2+\beta^2}$</td>
</tr>
<tr>
<td>damped sine</td>
<td>$e^{-\alpha t}\sin(\beta t)u(t)$</td>
<td>$\frac{\beta}{(s+\alpha)^2+\beta^2}$</td>
</tr>
<tr>
<td>damped cosine</td>
<td>$e^{-\alpha t}\cos(\beta t)u(t)$</td>
<td>$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$</td>
</tr>
</tbody>
</table>
Laplace Transform Properties

Linearity – absolutely critical property

Follows from the integral definition

\[ \mathcal{L}\{ Af_1(t) + Bf_2(t) \} = A \mathcal{L}\{ f_1(t) \} + B \mathcal{L}\{ f_2(t) \} = AF_1(s) + BF_2(s) \]

Example

\[ \mathcal{L}(A \cos(\beta t)) = \mathcal{L}\left( \frac{A}{2} \left[ e^{j\beta t} + e^{-j\beta t} \right] \right) = \frac{A}{2} \mathcal{L}(e^{j\beta t}) + \frac{A}{2} \mathcal{L}(e^{-j\beta t}) \]

\[ = \frac{A}{2} \frac{1}{s - j\beta} + \frac{A}{2} \frac{1}{s + j\beta} \]

\[ = \frac{As}{s^2 + \beta^2} \]
Laplace Transform Properties

Integration property

\[ \mathcal{L}\left\{ \int_{0}^{t} f(\tau) d\tau \right\} = \frac{F(s)}{s} \]

Proof

\[ \mathcal{L}\left\{ \int_{0}^{t} f(\tau) d\tau \right\} = \int_{0}^{\infty} \left[ \int_{0}^{t} f(\tau) d\tau \right] e^{-st} dt \]

Denote

\[ x = \frac{-e^{-st}}{s}, \text{ and } y = \int_{0}^{t} f(\tau) d\tau \]

so

\[ \frac{dx}{dt} = e^{-st}, \text{ and } \frac{dy}{dt} = f(t) \]

Integrate by parts

\[ \mathcal{L}\left\{ \int_{0}^{t} f(\tau) d\tau \right\} = \left[ \frac{-e^{-st}}{s} \int_{0}^{t} f(\tau) d\tau \right]_{0}^{\infty} + \frac{1}{s} \int_{0}^{\infty} f(t)e^{-st} dt \]
Laplace Transform Properties

Differentiation Property

\[ \mathcal{L}\left\{ \frac{df(t)}{dt} \right\} = sF(s) - f(0-) \]

Proof via integration by parts again

\[ \mathcal{L}\left\{ \frac{df(t)}{dt} \right\} = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} \, dt = \left[ \frac{df(t)}{dt} e^{-st} \right]_{0^-}^{\infty} + s \int_{0^-}^{\infty} f(t)e^{-st} \, dt \]

\[ = sF(s) - f(0-) \]

Second derivative

\[ \mathcal{L}\left\{ \frac{d^2 f(t)}{dt^2} \right\} = \mathcal{L}\left\{ \frac{d}{dt} \left[ \frac{df(t)}{dt} \right] \right\} = s\mathcal{L}\left\{ \frac{df(t)}{dt} \right\} - \frac{df}{dt}(0-) \]

\[ = s^2F(s) - sf(0-) - f'(0-) \]
Laplace Transform Properties

General derivative formula

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s) - s^{m-1} f(0-) - s^{m-2} f'(0-) - \cdots - f^{(m)}(0-)$$

Translation properties

$s$-domain translation

$$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s + \alpha)$$

$t$-domain translation

$$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-\alpha s} F(s) \quad \text{for} \quad a > 0$$
Laplace Transform Properties

Initial Value Property

\[
\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s F(s)
\]

Final Value Property

\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)
\]

Caveats:

- Laplace transform pairs do not always handle discontinuities properly
- Often get the average value
- Initial value property no good with impulses
- Final value property no good with \(\cos, \sin\) etc
Rational Functions

We shall mostly be dealing with LTs which are rational functions – ratios of polynomials in \( s \)

\[
F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}
\]

\[
= K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}
\]

\( p_i \) are the poles and \( z_i \) are the zeros of the function

\( K \) is the scale factor or (sometimes) gain

A proper rational function has \( n \geq m \)

A strictly proper rational function has \( n > m \)

An improper rational function has \( n < m \)
A Little Complex Analysis

We are dealing with linear ccts

Our Laplace Transforms will consist of rational function (ratios of polynomials in $s$) and exponentials like $e^{-s\tau}$

These arise from

- discrete component relations of capacitors and inductors
- the kinds of input signals we apply
  - Steps, impulses, exponentials, sinusoids, delayed versions of functions

Rational functions have a finite set of discrete poles

$e^{-s\tau}$ is an entire function and has no poles anywhere

To understand linear cct responses you need to look at the poles – they determine the exponential modes in the response circuit variables.

Two sources of poles: the cct – seen in the response to Ics
  the input signal LT poles – seen in the forced response
A Little More Complex Analysis

A complex function is *analytic* in regions where it has no poles

Rational functions are analytic everywhere except at a finite number of isolated points, where they have poles of finite order

Rational functions can be expanded in a Taylor Series about a point of analyticity

\[ f(z) = f(a) + (z - a)f'(a) + \frac{1}{2!}(z - a)^2 f''(a) + \ldots \]

They can also be expanded in a Laurent Series about an isolated pole

\[ f(z) = \sum_{n=-N}^{-1} c_n (z - a)^n + \sum_{n=0}^{\infty} c_n (z - a)^n \]

General functions do not have \( N \) necessarily finite
Residues at poles

Functions of a complex variable with isolated, finite order poles have *residues* at the poles

Simple pole: residue \( \lim_{s \to a} (s - a)F(s) \)

Multiple pole: residue \( \lim_{s \to a} \frac{d^{m-1}}{(m-1)!} (s - a)^m F(s) \)

The residue is the \( c_{-1} \) term in the Laurent Series

**Cauchy Residue Theorem**

The integral around a simple closed rectifiable positively oriented curve (scroc) is given by \( 2\pi j \) times the sum of residues at the poles inside
Inverse Laplace Transforms – the Bromwich Integral

\[ \mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\alpha-j\infty}^{\alpha+j\infty} F(s)e^{st} \, ds \]

This is a contour integral in the complex \( s \)-plane

\( \alpha \) is chosen so that all singularities of \( F(s) \) are to the left of \( \text{Re}(s) = \alpha \)

It yields \( f(t) \) for \( t \geq 0 \)

The inverse Laplace transform is always a causal function

For \( t < 0 \), \( f(t) = 0 \)

Remember Cauchy’s Integral Formula

Counterclockwise contour integral =

\[ 2\pi j \times (\text{sum of residues inside contour}) \]
Inverse Laplace Transform Examples

Bromwich integral of \( F(s) = \frac{1}{s + \alpha} \)

\[
f(t) = \int_{\alpha - j\infty}^{\alpha + j\infty} \frac{1}{s + \alpha} e^{st} \, ds
\]

\[
= \begin{cases} 
  e^{-at} & \text{for } t \geq 0 \\
  0 & \text{for } t < 0 
\end{cases}
\]

On curve \( C_1 \)

\[
s = \alpha + re^{j\theta}, \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}, \quad r \to \infty
\]

For given \( \theta \) there is \( r \to \infty \) such that

\[
\text{Re}(s) = \alpha + r \cos \theta < 0
\]

\[
e^{st} = e^{\text{Re}(s)t} e^{j\text{Im}(s)t} \to 0 \text{ as } r \to \infty \text{ for } t > 0
\]

Integral disappears on \( C_1 \) for positive \( t \)}
Inverting Laplace Transforms

Compute residues at the poles

\[ \lim_{s \to a} (s - a)F(s) \]

\[ \frac{1}{(m-1)!} \lim_{s \to a} \frac{d^{m-1}}{ds^{m-1}}[(s - a)^m F(s)] \]

Example

\[ \frac{2s^2 + 5s}{(s+1)^3} = \frac{2(s+1)^2 + (s+1) - 3}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{(s+1)^3} \]

\[ \lim_{s \to -1} \frac{(s+1)^3(2s^2 + 5s)}{(s+1)^3} = -3 \]

\[ \lim_{s \to -1} \frac{d}{ds} \left[ \frac{(s+1)^3(2s^2 + 5s)}{(s+1)^3} \right] = 1 \]

\[ \frac{1}{2!} \lim_{s \to -1} \frac{d^2}{ds^2} \left[ \frac{(s+1)^3(2s^2 + 5s)}{(s+1)^3} \right] = 2 \]

\[ \mathcal{L}^{-1} \left[ \frac{2s^2 + 5s}{(s+1)^3} \right] = e^{-t}(2 + t - 3t^2)u(t) \]
Inverting Laplace Transforms

Compute residues at the poles

\[
\lim_{s \to a} (s - a)F(s)
\]

\[
\frac{1}{(m-1)!} \lim_{s \to a} \frac{d^{m-1}}{ds^{m-1}} [(s - a)^m F(s)]
\]

Bundle complex conjugate pole pairs into second-order terms if you want

\[
(s - \alpha - j\beta)(s - \alpha + j\beta) = \left[ s^2 - 2\alpha s + (\alpha^2 + \beta^2) \right]
\]

but you will need to be careful

Inverse Laplace Transform is a sum of complex exponentials

For circuits the answers will be real
Inverting Laplace Transforms in Practice

We have a table of inverse LTs

Write \( F(s) \) as a partial fraction expansion

\[
F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{1}{K} \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \frac{\alpha_1}{(s-p_1)} + \frac{\alpha_2}{(s-p_2)} + \frac{\alpha_3}{(s-p_3)} + \frac{\alpha_3}{(s-p_3)^2} + \frac{\alpha_3}{(s-p_3)^3} + \cdots + \frac{\alpha_q}{(s-p_q)}
\]

Now appeal to linearity to invert via the table

Surprise!

Nastiness: computing the partial fraction expansion is best done by calculating the residues
Example 9-12

Find the inverse LT of \( F(s) = \frac{20(s + 3)}{(s + 1)(s^2 + 2s + 5)} \)

\[
F(s) = \frac{k_1}{s + 1} + \frac{k_2}{s + 1 - j2} + \frac{k^*_2}{s + 1 + j2}
\]

\[
k_1 = \lim_{s \to -1} (s + 1)F(s) = \left. \frac{20(s + 3)}{s^2 + 2s + 5} \right|_{s = -1} = 10
\]

\[
k_2 = \lim_{s \to -1 + 2j} (s + 1 - 2j)F(s) = \left. \frac{20(s + 3)}{(s + 1)(s + 1 + 2j)} \right|_{s = -1 + 2j} = -5 - 5j = 5\sqrt{2}e^{j\frac{5\pi}{4}}
\]

\[
f(t) = \left[ 10e^{-t} + 5\sqrt{2}e^{(-1 + j2)t + j\frac{5\pi}{4}} + 5\sqrt{2}e^{(-1 - j2)t - j\frac{5\pi}{4}} \right] u(t)
\]

\[
= \left[ 10e^{-t} + 10\sqrt{2}e^{-t} \cos(2t + \frac{5\pi}{4}) \right] u(t)
\]
Not Strictly Proper Laplace Transforms

Find the inverse LT of \( F(s) = \frac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3} \)

Convert to polynomial plus strictly proper rational function

Use polynomial division

\[
F(s) = s + 2 + \frac{s + 2}{s^2 + 4s + 3}
\]

\[
= s + 2 + \frac{0.5}{s + 1} + \frac{0.5}{s + 3}
\]

Invert as normal

\[
f(t) = \left[ \frac{d \delta(t)}{dt} + 2\delta(t) + 0.5e^{-t} + 0.5e^{-3t} \right]u(t)
\]
Multiple Poles

Look for partial fraction decomposition

\[ F(s) = \frac{K(s - z_1)}{(s - p_1)(s - p_2)^2} = \frac{k_1}{s - p_1} + \frac{k_{21}}{s - p_2} + \frac{k_{22}}{(s - p_2)^2} \]

\[ Ks - Kz_1 = k_1(s - p_2)^2 + k_{21}(s - p_1)(s - p_2) + k_{22}(s - p_1) \]

Equate like powers of \( s \) to find coefficients

\[ k_1 + k_{21} = 0 \]
\[ -2k_1 p_2 - 2k_{21}(p_1 + p_2) + k_{22} = K \]
\[ k_1 p_2^2 + k_{12} p_1 p_2 - k_{22} p_1 = Kz_1 \]

Solve
Introductory $s$-Domain Cct Analysis

First-order RC cct

KVL  \[ v_S(t) - v_R(t) - v_C(t) = 0 \]

instantaneous for each $t$

Substitute element relations

\[ v_S(t) = V_A u(t), \quad v_R(t) = Ri(t), \quad i(t) = C \frac{dv_C(t)}{dt} \]

Ordinary differential equation in terms of capacitor voltage

\[ RC \frac{dv_C(t)}{dt} + v_C(t) = V_A u(t) \]

Laplace transform

\[ RC[sV_C(s) - v_C(0)] + V_C(s) = \frac{1}{s} V_A \]

Solve

\[ V_C(s) = \frac{V_A / RC}{s(s + 1/RC)} + \frac{v_C(0)}{s + 1/RC} \]

Invert LT

\[ v_C(t) = \left[ V_A \left( 1 - e^{-t/RC} \right) + v_C(0) e^{-t/RC} \right] u(t) \quad \text{Volts} \]
An Alternative $s$-Domain Approach

Transform the cct element relations

Work in $s$-domain directly

$$V_C(s) = \frac{1}{Cs} I_C(s) + \frac{v_C(0)}{s}$$

$$I_C(s) = sCV_C(s) - Cv_C(0)$$

OK since $\mathcal{L}$ is linear

Impedance + source

Admittance + source

KVL in $s$-Domain

$$sCRV_C(s) - CRv_C(0) + V_C(s) = \frac{1}{s} V_A$$
Time-varying inputs

Suppose \( v_S(t) = V_a \cos(\beta t) \), what happens?

KVL as before

\[
(RCs + 1) V_C(s) - RCv_C(0) = \frac{sVA}{s^2 + \beta^2}
\]

\[
V_C(s) = \frac{sVA/RC}{(s^2 + \beta^2)(s + 1/RC)} + \frac{v_C(0)}{s + 1/RC}
\]

Solve

\[
v_C(t) = \left[ \frac{VA}{\sqrt{1 + (\beta RC)^2}} \cos(\beta t + \theta) - \frac{VA}{1 + (\beta RC)^2} e^{-t/RC} + v_C(0) e^{-t/RC} \right] u(t)
\]