Continuous-time Systems

(AKA analog systems)
Recall course objectives

Main Course Objective:
Fundamentals of systems/signals interaction
(we’d like to understand how systems transform or affect signals)

Specific Course Topics:
- Basic test signals and their properties
- Systems and their properties
- Signals and systems interaction
  Time Domain: convolution
  Frequency Domain: frequency response
- Signals & systems applications:
  audio effects, filtering, AM/FM radio
- Signal sampling and signal reconstruction
II. CT systems and their properties

Goals

I. A first classification of systems and their models:
   A. Operator Systems: maps that act on signals
   B. Physical Systems: examples and ODE models

II. Classification of systems according to their properties:
   Homogeneity, time invariance, superposition, linearity, memory, BIBO stability, controllability, invertibility, ...
Systems

Systems accept *excitations or input signals* and produce *responses or output signals*

Systems are often represented by block diagrams

SISO = single input, single output
MIMO = multiple input, multiple output

**SISO system**

\[ x(t) \xrightarrow{H} y(t) \]

**MIMO system**

\[ x_1(t) \xrightarrow{H_1} y_1(t) \]
\[ x_2(t) \xrightarrow{H_1} \xrightarrow{H_2} \xrightarrow{H_3} \xrightarrow{H_4} y_2(t) \]
Operator systems acting on signals

Systems can be “operators” or “maps” that combine signals

This is a more usual situation in the “digital world.” However, some analog systems can also be described through maps.

Examples:
1-Algebraic operators
   - noise removal by *averaging*
   - motion detection through image *subtraction*
2-Geometric/point operators (interpolation)
   - image *rectification*
   - visual spatial effects: morphing, image transformation
   - cartography: maps of ellipsoidal/spherical bodies
3-Signal multiplication:
   - AM radio signal modulation before transmission

However, simple operations like these are not enough to capture all filtering effects on signals (more on this later)
Noise* removal by averaging

\[ x(t) + n_1(t) \]
\[ x(t) + n_2(t) \]
\[ x(t) + n_3(t) \]
\[ \sum n_i(t) \]
\[ x(t) + \frac{\sum n_i(t)}{3} \]

Noisy versions of the signal (noise is zero mean)

Period of underlying signal
must be known or estimated

Matlab script in webpage: averaging.m

(*) noise = infinite-energy signal
that takes random values
Motion detection by subtraction

\[ A_0(x,y) - A_1(x,y) \]

(this is a digital system example: here image signals are a function of discrete-space variables or pixels)
Geometric/point operators

E.g., fish-eye lenses (used in robotics) distort reality

input-output
test using a known image for camera calibration
(we’d like to find \( f \):

\[
A(x,y) \rightarrow f \rightarrow B(x,y)
\]

correction of a real image using the inferred \( f \)
Signal multiplication

AM signal modulation for signal transmission (used in AM radio)

\[ x(t) \times \cos(\omega_c t) \]
II. CT systems and their properties

Goals

I. A first classification of systems and their models:
   A. Operator Systems: maps that act on signals
   B. Physical Systems: ODE models and examples

II. Classification of systems according to their properties:
   Homogeneity, time invariance, superposition, linearity, memory, BIBO stability, invertibility, controllability
ODEs and state-space system models

\[ \frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{d^m U(t)}{dt^m} + \ldots + b_{m-1} \frac{dU(t)}{dt} + b_m U(t) \]

The coefficients \( a_i, b_i \) can be time-varying or constant. When independent of \( y(t) \), ODE is linear, otherwise ODE is nonlinear.

We will typically consider \( b_i = 0 \) for \( i = 0, \ldots, m-1 \)

To solve the ODE we need to fix some initial conditions

\[ y(t_0), \frac{dy}{dt}(t_0), \frac{d^2 y}{dt^2}(t_0), \ldots, \frac{d^{n-1} y}{dt^{n-1}}(t_0) \]

(*) the unknowns are the \( y(t) \) and its derivatives

\[ n \] dimensional ODEs(*) model many electro-mechanical systems

Dimensional ODEs(*) model many electro-mechanical systems.
ODEs and state-space system models

Given:

\[
\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{d^m U(t)}{dt^m} + \ldots + b_{m-1} \frac{dU(t)}{dt} + b_m U(t)
\]

Possible inputs: linear combination of \( U(t) \) and derivatives
(represent known variables, e.g., forces in a Mech. System)

Possible outputs: linear combination of \( y(t) \) and/or its derivatives (represent unknown variables we would like to determine)

\[
z_l(t) = c_{l,0} \frac{d^n y(t)}{dt^n} + \ldots + c_{l,n-1} \frac{dy(t)}{dt} + c_{l,n} y(t) \quad l = 1, \ldots, n
\]

When coefficients are independent of \( y(t) \), the output equations are linear; otherwise, they are nonlinear
Physical Systems (mechanical)

Newtonian motion

\[ \sum F = ma \quad \sum \tau = Ia \]

\[ M \frac{d^2 y(t)}{dt^2} = -k_f \frac{dy(t)}{dt} + U(t) \]

Input \( U(t) \)
Output: \( y(t) \)
Initial conditions \( y(t_0), y'(t_0) \)

Second-order linear system

\[ \frac{d^2 y(t)}{dt^2} + \frac{k_f}{M} \frac{dy(t)}{dt} = \frac{1}{M} U(t), y(t_0), y'(t_0) \]
Physical Systems (mechanical)

Mass-spring-damper

\[ M \frac{d^2 y(t)}{dt^2} + D \frac{dy(t)}{dt} + Ky(t) = U(t) \]

Input \( U(t) \)

Outputs \( y(t) \), \( y'(t) \), or combination

Initial conditions \( y(t_0), y'(t_0) \)

Second-order linear system

\[ \frac{d^2 y(t)}{dt^2} + \frac{D}{M} \frac{dy(t)}{dt} + \frac{K}{M} y(t) = \frac{1}{M} U(t), \quad y(t_0), \quad y'(t_0) \]
Physical Systems (mechanical)

Simple pendulum

\[
I \frac{d^2 \theta(t)}{dt^2} = LU(t) - MgL \sin(\theta)
\]

I=ML^2, moment of inertia
Input (force at the ball) \(U(t)\)
Output \(\theta(t)\)
Initial conditions \(\theta(t_0), \theta'(t_0)\)

Second-order nonlinear system (why?)

\[
\frac{d^2 \theta(t)}{dt^2} + \frac{g}{L} \sin(\theta) = \frac{1}{ML} U(t) \quad \theta(t_0), \theta'(t_0)
\]

Linearization for small \(\theta\) is \(\theta''(t) + gL^{-1} \theta(t) = (ML)^{-1} U(t)\)
Kirchhoff’s laws

Current law:
sum of (signed) currents at a “node” is zero
(node = electrical juncture of two or more devices)

Voltage law:
sum of (signed) voltages around a “loop” is zero
(loop = closed path passing through ordered sequence of nodes)

Circuit element laws:

Resistor: \[ iR = V_R \]

Capacitor: \[ i = C \frac{dV}{dt} \]

Inductor: \[ V = L \frac{di}{dt} \]
Objective: Find a model relating the input \( i \) and output \( V_C \)

2 nodes and 2 loops

Equation of upper node: \( i - i_1 - i_2 = 0 \)
Equation of left loop: \( V + V_1 = 0 \)
Equation of right loop: \( -V_1 + V_C = 0 \)

Circuit element equations: \( V_1 = i_1R \) \( C \frac{dV_C}{dt} = i_2 \)

Putting it all together: \( i = i_2 + i_1 \) \( i = C \frac{dV_C}{dt} + \frac{1}{R} V_1 \)
ODEs and state-space models

A state-space representation of an nth order ODE describing a physical system is obtained as follows:

State  
\[ x = (x_1, x_2, x_3, ..., x_n)^T = \begin{pmatrix} y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, ..., \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}^T \]

The state-space representation of the system is a system of first-order differential equations in the new variables \( x_1, x_2, x_3, ..., x_n \).

If the original nth order ODE is linear, then the state-space representation can be expressed in matrix form:

\[ \frac{dx}{dt} = Ax + Bu, \]
\[ x(t_0) \]

\[ A = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ -a_{n1} & -a_{n-1} & -a_{n-2} & ... & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \\ b_m \end{pmatrix} \]

A linear output equation is expressed as \[ z = Cx, \quad C = (c_{l,k}) \]

This can be generalized for several variables.
Why state-space models are used

State-space formulation allows to lump multiple variables in a single state vector \( x \)

Distillation column:
Hundreds of state variables
  Concentration and temp at each tray position
  Lots of structure
    Output of one tray is the input to the next
Several inputs
  Boiler power, reflux ratio, feed rate
Many outputs
  Some tray temperatures, final concentration
Why state-space models are used

Well suited for MIMO systems

MIMO and SISO systems have same form in state-space formulation

This allows for uniform treatment

- Analysis of system properties
- Linearization
- Simulation (matlab)

\[
\begin{align*}
\begin{pmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t)
\end{pmatrix} &=
\begin{pmatrix}
\frac{-k_f}{M_1} & 0 & 0 \\
0 & \frac{-k_f}{M_2} & 0 \\
-1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{pmatrix} +
\begin{pmatrix}
M_1^{-1} & 0 \\
0 & M_2^{-1}
\end{pmatrix}
\begin{pmatrix}
f_1(t) \\
f_2(t)
\end{pmatrix} \\
\begin{pmatrix}
v_1(t) \\
v_2(t) \\
d(t)
\end{pmatrix} &=
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{pmatrix}
\end{align*}
\]
Why do we linearize about equilibrium points

Unfortunately, there are no general formulas to solve nonlinear ODEs. Then we are forced to look for (1) particular solutions and (2) approximations to the solutions.

How can we find particular solutions to nonlinear ODEs?
Equilibrium points are always particular constant solutions.

How to approximate the solutions of a nonlinear ODE?
(a) We know how to solve linear ODEs.
(b) The qualitative behavior of a nonlinear ODE with an initial condition close to an equilibrium point, under inputs of small magnitude, can be found by solving the linearized equation about that equilibrium point with zero inputs.
Linearization about equilibrium point

\[
\frac{dx}{dt} = f(x,u), \quad x(t_0),
\]

An equilibrium point is \( x_0 \) such that \( f(x_0,0) = 0 \)

**Equilibrium point = constant solution to ODE**
The system remains at rest at all times if initially placed at the equilibrium and no inputs are applied

Pendulum example. The state is \( x = (\theta, \theta')^T \) The pendulum has two equilibrium points:

\[
x_1 = (0,0)^T \text{ (vertical bottom position, zero velocity)}
\]

\[
x_2 = (\pi,0)^T \text{ (vertical top position, zero velocity)}
\]
Linearization is easy in state-space formulation

Suppose the map \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is nonlinear (as in the pendulum example)

**Linearization** of the system about \( x_0 \) with \( u = 0 \) is:

\[
\frac{dx}{dt} = \left( \frac{\partial f}{\partial x} \right)_{x=x_0, u=0} (x - x_0) + \left( \frac{\partial f}{\partial u} \right)_{x=x_0, u=0} u, \quad x(t_0)
\]

with constant matrices

\[
\left( \frac{\partial f}{\partial x} \right)_{x=x_0, u=0} = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\cdots & \cdots & \cdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix}
\quad \text{and} \quad
\left( \frac{\partial f}{\partial u} \right)_{x=x_0, u=0} = \begin{pmatrix}
\frac{\partial f_1}{\partial u} \\
\cdots \\
\frac{\partial f_n}{\partial u}
\end{pmatrix}
\]
Linearization with additional output map

For systems with an additional nonlinear output map:

\[ \frac{dx}{dt} = f(x,u), \quad x(t_0), \]

\[ z = h(x), \]

where \( h : R^n \rightarrow R^p \), \( h(x_0) = 0 \), **linearization** becomes:

\[ \frac{dx}{dt} = \left( \frac{\partial f}{\partial x} \right)_{|x=x_0,u=0} (x - x_0) + \left( \frac{\partial f}{\partial u} \right)_{|x=x_0,u=0} u, \]

\[ z = \left( \frac{\partial h}{\partial x} \right)_{|x=x_0} (x - x_0) \]
CT systems and their properties

Goals

I System examples and their models e.g. using basic principles
   A. Operator systems: maps that act on signals
   B. Physical systems: ODE models and examples

II System properties
   Homogeneity, time invariance, superposition, linearity, memory, invertibility, BIBO stability, controllability
Response of a RC Low-pass filter

An RC low-pass filter is a simple circuit

It can be modeled as a single input, single output system

The system is excited by a voltage \( v_{in}(t) \) and responds with a voltage \( v_{out}(t) \)

We assume the circuit has no initial energy at the capacitor
Response of a RC Low-pass filter

If the RC low-pass filter is excited by a step voltage

\[ v_{in}(t) = A u(t) \]

Its response is

\[ v_{out}(t) = A \left( 1 - e^{-t/RC} \right) u(t) \]

That is, if the excitation is doubled, the response doubles.
Homogeneity

In a **homogeneous** system, multiplying the excitation by any constant (including complex constants), multiplies the response by the same constant.
Homogeneity Test

Homogeneity Test:
1) apply an arbitrary $g(t)$ as input and obtain $y_1(t)$ as output
2) then apply $Kg(t)$ and obtain its output, $h(t)$

If $h(t) = Ky_1(t)$ then the system is homogeneous
Time invariance

If an excitation causes a response and delaying the excitation simply delays the response by the same amount of time, then the system is **time invariant**

\[
\text{Time Invariant System}
\]

\[
x(t) \xrightarrow{\mathcal{H}} y(t)
\]

\[
x(t) \xrightarrow{\text{Delay, } t_0} x(t - t_0) \xrightarrow{\mathcal{H}} y(t - t_0)
\]

If \( g(t) \xrightarrow{\mathcal{H}} y_1(t) \) and \( g(t - t_0) \xrightarrow{\mathcal{H}} y_1(t - t_0) \) \( \Rightarrow \mathcal{H} \) is Time Invariant

This test must succeed for any \( g \) and any \( t_0 \).
Additivity property

If one excitation causes a response and another excitation causes another response and if, for any arbitrary excitations, the sum of the two excitations causes a response which is the sum of the two responses, the system is said to be additive.

\[ g(t) \xrightarrow{\mathcal{H}} y_1(t) \text{ and } h(t) \xrightarrow{\mathcal{H}} y_2(t) \]

and \( g(t) + h(t) \xrightarrow{\mathcal{H}} y_1(t) + y_2(t) \Rightarrow \mathcal{H} \text{ is Additive} \)
Linearity and LTI systems

If a system is both homogeneous and additive, it is **linear**

If a system is both linear and time-invariant, it is called an **LTI (linear, time-invariant)** system

Some systems which are **non-linear** can be **accurately approximated** for analytical purposes by a **linear system** for small excitations (recall the discussion on linearization)

We will mainly focus on LTI systems because we can characterize their response to any signal
System Invertibility

A system is invertible if unique excitations produce unique responses.

In other words, in an invertible system, knowledge of the response is sufficient to determine the excitation.

Any system with input $x(t)$ and output $y(t)$ just described by a linear ODE of the form

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = x(t)$$

is invertible. A system with input $x(t)$ and output $z(t)$ described by the operator map $z(t) = \sin(y(t))$ is non-invertible because $\sin(y)$ does not have an inverse.
Memory

This concept reflects the extent to which the present behavior of a system (its outputs) is affected by its past (initial conditions or past values of the inputs).

Physical systems modeled through ODEs have memory: this is associated with the system inability to dissipate energy or redistribute it instantaneously.

Example: Think about how a pendulum initially off the vertical winds down to the equilibrium position. The time it takes to do it captures the pendulum memory.

If a system is well understood then one can relate its memory to specific properties of the system (e.g. “system stability”).

In fact, all filtering methods in signal processing are based on exploiting the memory properties of systems.
Memory

A system is said to be **memoryless** if for any time $t_1$, the output at $t_1$ depends only on the input at time $t_1$.

Example:
- If $y(t) = K u(t)$, then the system is memoryless.
- If $y(t) = K u(t-1)$, then it has memory.

(Operator systems described through static maps are usually memoryless.)

Any system that contains a derivative in it has memory; e.g., any system described through an ODE.
Stability

Any system for which the response is bounded for any arbitrary bounded excitation is said to be bounded-input-bounded-output (BIBO) stable system, otherwise it is unstable.

**Intuition:** All systems for which outputs “do not explode” (i.e. outputs can only reach finite values) are BIBO stable. Intuitively, if a system has a “small memory” or “dissipates energy quickly,” then it will be stable.

If an ODE describing the system is available, then we can apply the following test for BIBO stability:

A system described by a differential equation is stable if the eigenvalues of the solution of the equation all have negative real parts.
Stability

Stable systems return to equilibrium despite input disturbances
How to check Stability when ODEs available

Suppose a state-space model for the physical system is available

\[
\frac{dx}{dt} = Ax + Bg, \quad x(t_0)
\]

Then, the system is stable if and only if the eigenvalues of the matrix \( A \) (= the eigenvalues of the ODE) have all negative real parts

The eigenvalues of \( A \) are the solutions \( \lambda \) to the equation

\[
\det(\lambda I_n - A) = 0
\]

Here, \( n \) is the dimension of the state \( x \)
How to check Stability when ODEs available

Simple example: Low-pass filter

ODE:

\[
\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} g(t)
\]

state-space representation:

\[
\frac{dy(t)}{dt} = -\frac{1}{RC} y(t) + \frac{1}{RC} g(t)
\]

Matrix \( A \) is just a number:

\[
A = -\frac{1}{RC}
\]

Calculation of eigenvalues:

\[
\det(\lambda \times 1 + \frac{1}{RC}) = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{RC}
\]

Thus, the system is BIBO stable
System controllability

A physical system described by a linear state-space model

\[ \frac{dx}{dt} = Ax + Bg, \quad x(t_0) \]

is **controllable** if and only if for any initial condition \( x(t_0) \) there exists a control \( g(t) \) so that we can reach any final state \( x(t_f) \) after finite time. That is, if the system is controllable we can do anything with it!

Controllability Theorem: A linear system is controllable if and only if

\[ \text{rank} [B, AB, A^2B, A^3B, ..., A^{n-1}B] = n \]

\( n \) is the dimension of the state of the system

Control how? The **Controllability Grammian** is used to find the right \( g(t) \) (out of the scope of this course)
Summary

Important points to remember:

1. We can model simple (mechanical/electric) systems by resorting to basic principles and producing ODEs.

2. A special system representation is the state-space representation, useful for simulation, linearization and to check system controllability.

3. Special system properties are homogeneity, additivity, time-invariance, LTI, invertibility, memory, and stability. These properties can be checked by looking at input-output experiments (no models required in principle.)

4. If an ODE model of the system is available, we can check system stability by finding the eigenvalues of the ODE.

5. If an state space representation of the system is available, we can check the system controllability properties by applying the controllability theorem.