1 Some Facts on Symmetric Matrices

Definition: Matrix $A$ is symmetric if $A = A^T$.

Theorem: Any symmetric matrix
1) has only real eigenvalues;
2) is always diagonalizable;
3) has orthogonal eigenvectors.

Corollary: If matrix $A$ then there exists $Q^TQ = I$ such that $A = Q^T \Lambda Q$.

Proof:
1) Let $\lambda \in \mathbb{C}$ be an eigenvalue of the symmetric matrix $A$. Then $Av = \lambda v$, $v \neq 0$, and
   
   \[ v^*Av = \lambda v^*v, \quad v^* = \bar{v}^T. \]

   But since $A$ is symmetric
   
   \[ \lambda v^*v = v^*Av = (v^*Av)^* = \bar{\lambda}v^*v. \]

   Therefore, $\lambda$ must be equal to $\bar{\lambda}$!

2) If the symmetric matrix $A$ is not diagonalizable then it must have
   generalized eigenvalues of order 2 or higher. That is, for some repeated
   eigenvalue $\lambda_i$ there exists $v \neq 0$ such that
   
   \[ (A - \lambda_i I)^2v = 0, \quad (A - \lambda_i I)v \neq 0 \]

   But note that
   
   \[ 0 = v^*(A - \lambda_i I)^2v = v^*(A - \lambda_i I)(A - \lambda_i I) \neq 0, \]

   which is contradiction. Therefore, as there exists no generalized eigenvectors
   of order 2 or higher, $A$ must be diagonalizable.

3) As $A$ must have no generalized eigenvector of order 2 or higher
   
   $AT = A \left[ v_1 \cdots v_n \right] = \left[ v_1 \cdots v_n \right] \Lambda = T\Lambda, \quad |T| \neq 0.$

   That is $A = T^{-1}\Lambda T$. But since $A$ is symmetric
   
   $T^{-1}\Lambda T = A = A^T = (T^{-1}\Lambda T)^T = T^T\Lambda T^{-T} \quad \Rightarrow \quad T^T = T^{-1}$

   or
   
   $T^T T = I \quad \Rightarrow \quad v_i^T v_i = 1, \quad v_i^T v_j = 0, \forall i \neq j.$
1.1 Positive definite matrices

**Definition:** The symmetric matrix $A$ is said positive definite ($A > 0$) if all its eigenvalues are positive.

**Definition:** The symmetric matrix $A$ is said positive semidefinite ($A \geq 0$) if all its eigenvalues are non-negative.

**Theorem:** If $A$ is positive definite (semidefinite) there exists a matrix $A^{1/2} > 0$ ($A^{1/2} \geq 0$) such that $A^{1/2}A^{1/2} = A$.

**Proof:** As $A$ is positive definite (semidefinite)

\[
A = Q^T \Lambda Q, \quad Q^T Q = QQ^T = I \\
= Q^T \Lambda^{1/2} \Lambda^{1/2} Q, \quad \Lambda^{1/2}_{ii} = \sqrt{\lambda_i} \\
= \underbrace{Q^T \Lambda^{1/2} Q}_{A^{1/2}} \underbrace{Q^T \Lambda^{1/2} Q}_{A^{1/2}},
\]

**Theorem:** $A$ is positive definite if and only if $x^T Ax > 0, \quad \forall x \neq 0$.

**Proof:**

Assume there is $x \neq 0$ such that $x^T Ax \leq 0$ and $A$ is positive definite. Then there exists $Q^T Q = I$ such that $A = Q^T \Lambda Q$ with $\Lambda_{ii} = \lambda_i > 0$. Then for $y \neq 0$ such that $x = Q^T y$

\[
0 \geq x^T Ax = y^T QAQy = y^T QQ^T \Lambda QQ^T y = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2 \geq 0
\]

which is a contradiction.
2 Controllability Gramian

LTI system in state space
\[
\dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t)
\]

**Problem:** Given \( x(0) = 0 \) and any \( \bar{x} \), compute \( u(t) \) such that \( x(\bar{t}) = \bar{x} \) for some \( \bar{t} > 0 \).

**Solution:** We know that
\[
\bar{x} = x(\bar{t}) = \int_{0}^{\bar{t}} e^{A(\bar{t} - \tau)} Bu(\tau) d\tau.
\]

If we limit our search for solutions \( u \) in the form
\[
u(t) = B^{T} e^{A^{T}(\bar{t} - t)} \bar{z}
\]
we have
\[
\bar{x} = \int_{0}^{\bar{t}} e^{A(\bar{t} - \tau)} BB^{T} e^{A^{T}(\bar{t} - \tau)} \bar{z} d\tau,
\]
\[
= \left( \int_{0}^{\bar{t}} e^{A(\bar{t} - \tau)} BB^{T} e^{A^{T}(\bar{t} - \tau)} d\tau \right) \bar{z}, \quad \xi = \bar{t} - \tau
\]
\[
= \left( \int_{0}^{\bar{t}} e^{A\xi} BB^{T} e^{A^{T}\xi} d\xi \right) \bar{z},
\]
and
\[
\bar{z} = \left( \int_{0}^{\bar{t}} e^{A\xi} BB^{T} e^{A^{T}\xi} d\xi \right)^{-1} \bar{x},
\]
\[
\Rightarrow \quad u(t) = B^{T} e^{A^{T}(\bar{t} - t)} \left( \int_{0}^{\bar{t}} e^{A\xi} BB^{T} e^{A^{T}\xi} d\xi \right)^{-1} \bar{x}
\]
The symmetric matrix
\[
X(t) := \int_{0}^{t} e^{A\xi} BB^{T} e^{A^{T}\xi} d\xi
\]
is known as the **Controllability Gramian**.

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2.1 Properties of the Controllability Gramian

**Theorem:** The Controllability Gramian

\[ X(t) = \int_{0}^{t} e^{A\xi} BB^{T} e^{A^{T}\xi} d\xi, \]

is the solution to the differential equation

\[ \frac{d}{dt} X(t) = AX(t) + X(t)A^{T} + BB^{T}. \]

If \( X = \lim_{t \to \infty} X(t) \) exists then

\[ AX + XA^{T} + BB^{T} = 0. \]

**Proof:** For the first part, compute

\[
\frac{d}{dt} X(t) = \frac{d}{dt} \int_{0}^{t} e^{A\xi} BB^{T} e^{A^{T}\xi} d\xi = \frac{d}{dt} \int_{0}^{t} e^{A(t-\tau)} BB^{T} e^{A^{T}(t-\tau)} d\tau, \\
= \int_{0}^{t} \frac{d}{d\tau} e^{A(t-\tau)} BB^{T} e^{A^{T}(t-\tau)} + e^{A(t-\tau)} BB^{T} e^{A^{T}(t-\tau)} \big|_{\tau=t}, \\
= A \left( \int_{0}^{t} e^{A(t-\tau)} BB^{T} e^{A^{T}(t-\tau)} d\tau \right) \\
+ \left( \int_{0}^{t} e^{A(t-\tau)} BB^{T} e^{A^{T}(t-\tau)} d\tau \right) A^{T} + BB^{T}, \\
= AX(t) + X(t)A^{T} + BB^{T}. 
\]

For the second part, use the fact that \( X(t) \) is smooth and therefore

\[
\lim_{t \to \infty} X(t) = X \quad \Rightarrow \quad \lim_{t \to \infty} \frac{d}{dt} X(t) = 0.
\]
2.2 Summary on Controllability

**Theorem:** The following are equivalent

1) The pair \((A, B)\) is controllable;

2) The Controllability Matrix \(C(A, B)\) has full-row rank;

3) There exists no \(z \neq 0\) such that \(z^* A = \lambda z, \quad z^* B = 0\);

4) The Controllability Gramian \(X(t)\) is positive definite for some \(t \geq 0\).

**Proof:**
Everything has already been proved except the equivalence of 4).

* Sufficiency: Immediate from the construction of \(u(t)\).

* Necessity: First part:

\[
X(t) = \int_0^t e^{A\xi} B B^T e^{A^T \xi} d\xi \geq 0
\]

by construction. We have to prove that when \((A, B)\) is controllable then \(X(t) > 0\). To prove this assume that \((A, B)\) is controllable but \(X(t)\) is not positive definite. So there exists \(z \neq 0\) such that

\[
z^* e^{A\tau} B = 0, \quad \forall 0 \leq \tau \leq t.
\]

But this implies

\[
\frac{d^i}{d\tau^i}(i! \, z^* e^{A\tau} B) \bigg|_{\tau=0} = z^* A^i e^{A\tau} B \bigg|_{\tau=0} = z^* A^i B = 0, \quad i = 0, \ldots, n - 1
\]

which implies \(C(A, B)\) does not have full-row rank (see proof of the Popov-Belevitch-Hautus Test).
3 Observability Gramian

LTI system in state space

\[
\dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t)
\]

**Problem:** Given \(u(t) = 0\) and \(y(t)\) compute \(x(0)\).

**Solution:** We know that

\[y(t) = C e^{At} x(0).\]

Multiplying on the left by \(e^{AT} C^T\) and integrating from 0 to \(t\) we have

\[
\int_0^t e^{AT} C^T y(\xi) d\xi = \left( \int_0^t e^{AT} C^T C e^{A\xi} d\xi \right) x(0)
\]

from which

\[
x(0) = \left( \int_0^t e^{AT} C^T C e^{A\xi} d\xi \right)^{-1} \int_0^t e^{AT} C^T y(\xi) d\xi.
\]

The symmetric matrix

\[
Y(t) := \int_0^t e^{AT} C^T C e^{A\xi} d\xi
\]

is known as the **Observability Gramian**.
3.1 Properties of the Observability Gramian

**Theorem:** The Observability Gramian

\[ Y(t) = \int_0^t e^{A_T \xi} C^T C e^{A \xi} d\xi, \]

is the solution to the differential equation

\[ \frac{d}{dt} Y(t) = A^T Y(t) + Y(t) A + C^T C. \]

If \( Y = \lim_{t \to \infty} X(t) \) exists then

\[ A^T Y + YA + C^T C = 0. \]

3.2 Summary on Observability

**Theorem:** The following are equivalent

1) The pair \((A, C)\) is observable;

2) The Observability Matrix \( \mathcal{O}(A, C) \) has full-column rank;

3) There exists no \( x \neq 0 \) such that \( Ax = \lambda x, \quad Cx = 0; \)

4) The Observability Gramian \( Y = Y(t) \) is positive definite for some \( t \geq 0. \)
**Lemma:** Consider the Lyapunov Equation

\[ A^T X + XA + C^T C = 0 \]

where \( A \in \mathbb{C}^{n \times n} \) and \( C \in \mathbb{C}^{m \times n} \).

1. A solution \( X \in \mathbb{C}^{n \times n} \) exists and is unique if and only if \( \lambda_j(A) + \lambda_i^*(A) \neq 0 \) for all \( i, j = 1, \ldots, n \). Furthermore \( X \) is symmetric.

2. If \( A \) is Hurwitz then \( X \) is positive semidefinite.

3. If \( (A, C) \) is detectable and \( X \) is positive semidefinite then \( A \) is Hurwitz.

4. If \( (A, C) \) is observable and \( A \) is Hurwitz then \( X \) is positive definite.

**Proof:**

Item 1. The Lyapunov Equation is a linear equation and it has a unique solution if and only if the homogeneous equation associated with the Lyapunov equation admits only the trivial solution. Assume it does not, that is, there \( \bar{X} \neq 0 \) such that

\[ A^T \bar{X} + \bar{X}A = 0 \]

Then, multiplication of the above on the right by \( x_i^* \neq 0 \), the \( i \)th eigenvector of \( A \) and on the right by \( x_j^* \neq 0 \) yields

\[ 0 = x_i^* A^T \bar{X} x_j + x_j^* \bar{X} A x_i = [\lambda_j(A) + \lambda_i^*(A)] x_i^* \bar{X} x_j. \]

Since \( \lambda_i(A) + \lambda_j(B) \neq 0 \) by hypothesis we must have \( x_i^* \bar{X} x_j = 0 \) for all \( i, j \).

One can show that this indeed implies \( \bar{X} = 0 \), establishing a contradiction. That \( X \) is symmetric follows from uniqueness since

\[ 0 = (A^T X + XA + C^T C)^T - (A^T X + XA + C^T C) \]
\[ = A^T (X^T - X) + (X^T - X) A \]

so that \( X^T - X = 0 \).

Item 2. If \( A \) is Hurwitz then \( \lim_{t \to \infty} e^{At} = 0 \). But

\[ X = \int_0^\infty e^{A^T t} C^T C e^{At} dt \geq 0 \]

and

\[ A^T X + XA = \lim_{t \to \infty} \int_0^\infty \frac{d}{dt} e^{A^T t} C^T C e^{At} dt = e^{A^T t} C^T C e^{At} \bigg|_0^\infty = -C^T C. \]
4 Controllability, Observability and Duality

Primal LTI system in state space

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t)
\end{align*}
\]

Dual LTI system in state space

\[
\begin{align*}
\dot{x}(t) &= A^T x(t) + C^T u(t), \\
y(t) &= B^T x(t).
\end{align*}
\]

The primal system is observable if and only if the dual system is controllable. The primal system is controllable if and only if the dual system is observable.