1 Stability

1.1 Bounded Input-Bounded Output (BIBO) Stability

**Definition:** A system \( y = Hu \) is BIBO stable if for any bounded input \( u(t) \) corresponds a bounded output \( y(t) \).

In general, the input \( u(t) \) and the output \( y(t) \) are bounded in the sense of a signal norm! A scalar signal \( u(t) \) is bounded if
\[
\exists M_u < \infty : \|u(t)\| = \sup_{t \geq 0} |u(t)| < M_u.
\]

**Definition:** A scalar function \( h(t) \) is absolutely integrable if \( \exists M_h < \infty \) such that
\[
\int_0^\infty |h(\tau)| d\tau < M_h.
\]

**Theorem:** The SISO linear system with impulse response \( h(t) \) is BIBO stable if and only if \( h(t) \) is absolutely integrable.

**Theorem:** The MIMO linear system with impulse response matrix \( H(t) = (H_{ij}(t)) \) is BIBO stable if and only if \( h_{ij}(t) \) is absolutely integrable for all \( i, j \).
Corollary: The MIMO linear system with a rational and proper transfer matrix $H(s) = N(s)/d(s)$ is BIBO stable if and only if all poles of $H(s)$, i.e., the roots of $d(s)$, are in the open left-half of the complex plane.

Proof: The impulse response $h_{ij}(t)$ can be obtained from $H_{ij}(s)$ as

$$h_{ij}(t) = \mathcal{L}^{-1}\left\{H_{ij}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{N_{ij}(s)}{d(s)}\right\}.$$  

Expanding $H_{ij}(s)$ in partial fractions we have

$$h_{ij}(t) = \mathcal{L}^{-1}\left\{\sum_{i=1}^{m} \sum_{j=1}^{k_i} \frac{\alpha_i}{(s - \lambda_i)^j}\right\} = \sum_{i=1}^{m} \sum_{j=1}^{k_i} t^{j-1} e^{\lambda_i t}$$

where $\lambda_i$ denotes the $i$th root of $d(s)$ with multiplicity $k_i$. Therefore, $h_{ij}(t)$ is absolutely integrable if and only if the factors $t^{j-1} e^{\lambda_i t}$, $j = 1, \ldots, k_i$ are absolutely integrable. That is, iff $\Re(\lambda_i) < 0$. 


Proof of Theorem (SISO): Sufficiency: For linear systems

\[ y(t) = \int_0^\infty h(\tau)u(t - \tau) \, d\tau \]

and for SISO systems

\[ |y(t)| = \left| \int_0^\infty h(\tau)u(t - \tau) \, d\tau \right|, \]
\[ \leq \int_0^\infty |h(\tau)u(t - \tau)| \, d\tau, \]
\[ \leq \int_0^\infty |h(\tau)||u(t - \tau)| \, d\tau. \]

Therefore, since \( u(t) \) is bounded and \( h(t) \) is absolutely integrable

\[ |y(t)| \leq \int_0^\infty |h(\tau)||u(t - \tau)| \, d\tau, \]
\[ \leq M_u \int_0^\infty |h(\tau)| \, d\tau, \]
\[ \leq M_u M_h \]

which implies \( y(t) \) is bounded.

Necessity: If \( h(t) \) is not absolutely integrable then there exists \( \bar{\epsilon} \) such that

\[ \int_{0}^{\bar{\epsilon}} |h(\tau)| \, d\tau = \infty. \]

Now for the particular input

\[ u(\bar{\epsilon} - t) = \begin{cases} +1, & h(t) \geq 0 \\ -1, & h(t) < 0 \end{cases} \]

we have that

\[ y(\bar{\epsilon}) = \int_0^{\bar{\epsilon}} h(\tau)u(\bar{\epsilon} - \tau) \, d\tau = \int_0^{\bar{\epsilon}} |h(\tau)| \, d\tau = \infty \]

which implies \( y(t) \) is not bounded.
1.2 Internal Stability

The LTI system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t).
\end{align*}
\]  

is BIBO stable iff \( H(s) = C(sI - A)^{-1}B + D \) has all poles on the open left-half of the complex plane.

The LTI system (1) is \textit{internally stable} iff all roots of \( d(s) = \det(sI - A) \) are on the open left-half of the complex plane.

\[
\text{Internal stability} \quad \implies \quad \text{BIBO stability}
\]

\[
\text{Internal stability} \quad \iff \quad \text{BIBO stability} + \quad \text{controllability and observability}
\]
1.3 Lyapunov Stability

Consider the dynamic system
\[ \dot{x} = f(x). \]

**Definition:** \( x \) is an equilibrium point of (1.3) if \( f(x) = 0 \).

**Definition:** Let \( x = 0 \) be an equilibrium point of (1.3) and let \( \Omega \subset \mathbb{R}^n \). It is
1) **Stable** if for each \( \epsilon > 0 \) there is \( \delta > 0 \) such that
   \[ \|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq 0. \]
2) **Unstable** if not stable.
3) **Asymptotically stable** if it is stable and \( \delta > 0 \) can be chosen such that
   \[ \|x(0)\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0. \]

**Theorem:** Let \( x = 0 \) be an equilibrium point of the dynamic system (1.3) and \( x \in \Omega \subset \mathbb{R}^n \). Let \( V : \Omega \to \mathbb{R} \) be continuously differentiable and
\[ V(0) = 0, \quad V(x) > 0 \quad \forall x \in \Omega, \quad x \neq 0, \]
and
\[ \dot{V}(x) \leq 0, \forall x \in \Omega. \]
Then \( x = 0 \) is **stable**. Moreover, if
\[ \dot{V}(x) < 0, \forall x \in \Omega, x \neq 0 \]
then \( x = 0 \) is **asymptotically stable**.

**Proof:** see H. K. Kalil, *Nonlinear Systems*, Chapter 3.
1.3.1 Lyapunov Stability for Linear Systems

Consider the LTI system
\[ \dot{x} = Ax. \] (2)

**Theorem:** The following statements about the linear system (2) are equivalent:

1) \( \Re(\lambda_i(A)) < 0. \)
2) \( x = 0 \) is the unique equilibrium point of (2) and it is asymptotically stable.

**Proof:** Use the same tools to conclude that
\[ x(t) = \mathcal{L} \{ (sI - A)^{-1} x(0) \} \]
so that
\[ \lim_{t \to \infty} x(t) = 0 \]
for any \( x(0) \) bounded iff \( \Re(\lambda_i(A)) < 0. \) This implies \( A \) is nonsingular, which implies that \( Ax = 0 \Rightarrow x = 0 \) is the unique equilibrium point.
Theorem: The LTI system (2) is asymptotically stable if and only if, for any matrix $Q > 0$, the Lyapunov equation

\[ A^T P + PA + Q = 0 \]  

has a unique solution $P$ such that $P > 0$.

Proof: Sufficiency: Assume there exists $P > 0$ that solves (3). Then $V(x) = x^T P x > 0$, for all $0 \neq x \in \Omega := \mathbb{R}^n$. Note that

\[
\dot{V}(x) = \frac{d}{dt} x^T P x, \\
= \dot{x}^T P x + x^T P \dot{x}, \\
= x^T (A^T P + PA) x, \\
= -x^T Q x < 0, \quad \forall 0 \neq x \in \mathbb{R}^n.
\]

Therefore (2) is asymptotically stable.

Necessity: Assume (2) is asymptotically stable then $\mathbb{R}(\lambda_i(A)) < 0$. This can be used to show that the Gramian

\[
P = \int_0^\infty e^{A^T t} Q e^{A t} dt
\]

converges to a finite value. Using stability we can also show that

\[
\lim_{t \to \infty} e^{A^T t} Q e^{A t} = 0.
\]

Therefore, as we already know

\[
A^T P + PA = A^T \left( \int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{A t} dt \right) + \left( \int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{A t} dt \right) A, \\
= \int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{A t} dt, \\
= \lim_{t \to \infty} (e^{A^T t} Q e^{A t}) - Q = -Q.
\]

To prove that it is positive definite we assume that it is not, so that there exists $z \neq 0$ such that (see a similar proof for the Controllability and Observability Gramians)

\[
z^* P = z^* \int_0^\infty e^{A^T t} Q e^{A t} dt = 0 \quad \Rightarrow \quad e^{A t} z \equiv 0, \quad \forall \ t \geq 0 \quad \Rightarrow \quad z = 0.
\]
Finally, to prove that it is unique assume there exists \( \tilde{P} \neq P \) satisfying the Lyapunov equation. Then

\[
A^T(P - \tilde{P}) + (P - \tilde{P})A = 0
\]

and

\[
e^{A^Tt} \left[ A^T(P - \tilde{P}) + (P - \tilde{P})A \right] e^{At} = A^T \left[ e^{A^Tt}(P - \tilde{P})e^{At} \right] + \left[ e^{A^Tt}(P - \tilde{P})e^{At} \right] A,
\]

\[
= \frac{d}{dt} e^{A^Tt}(P - \tilde{P})e^{At} = 0, \quad \forall t \geq 0,
\]

which implies that

\[
e^{A^Tt}(P - \tilde{P})e^{At} \text{ is constant} \quad \forall t \geq 0.
\]

In particular, at \( t = 0 \) and \( t \to \infty \)

\[
(P - \tilde{P}) = \lim_{t \to \infty} e^{A^Tt}(P - \tilde{P})e^{At} = 0 \quad \Rightarrow \quad \tilde{P} = P
\]

**REMARK #1:** The proof works also if \( Q = C^TC \geq 0 \) and \((A, C)\) is observable.

**REMARK #2:** Internal stability \( \iff \) Asymptotic stability.
1.4 Example

\[ \dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x \]

Lyapunov Equation \((Q = I)\)

\[
\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

is equivalent to the linear system

\[
\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}
\]

which has the unique solution

\[
\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} > 0
\]