1 Linear Time-Varying Systems

LTV system in state space

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \\
y(t) = C(t)x(t) + D(t)u(t).
\]

1.1 Existence and uniqueness of solution

Differential equation:

\[
\dot{x}(t) = f(x(t), t), \quad a \leq t \leq b.
\]

Sufficient condition for existence and uniqueness of solution: \( f(x, t) \) is Lipschitz, i.e.,

\[
\|f(y(t), t) - f(x(t), t)\| \leq k(t)\|y(t) - x(t)\|, \quad a \leq t \leq b,
\]

where \( k() \) is (piecewise) continuous.

For LTV \( f(x, t) = A(t)x(t) + B(t)u(t) \) and

\[
\|f(y(t), t) - f(x(t), t)\| = \|A(t)y(t) + B(t)u(t) - A(t)x(t) - B(t)u(t)\|,
\]
\[
= \|A(t)y(t) - A(t)x(t)\|,
\]
\[
\leq \|A(t)\|\|y(t) - x(t)\| \quad \Rightarrow \quad \text{Lipschitz!}
\]

Conclusion: LTV system has a solution and it is unique!
1.2 Solution to LTV

Scalar homogeneous equation

\[ \dot{x}(t) = a(t)x(t), \quad t \geq 0, \quad x(0) = x_0. \]

Separation of variables

\[ \frac{1}{x} dx = a(t) dt. \]

Integrate on both sides

\[ \int_{x(0)}^{x(t)} \frac{1}{x} dx = \int_0^t a(\tau) d\tau, \quad \Rightarrow \quad x(t) = e^{\int_0^t a(\tau) d\tau} x(0). \]

In the matrix case, assume

\[ x(t) = e^{\int_0^t A(\tau) d\tau} x(0), \]

and compute

\[
\frac{d}{dt} x(t) = \frac{d}{dt} e^{F(t)} x(0), \quad F(t) = \int_0^t A(\tau) d\tau, \\
= \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d}{dt} F^i(t) x(0).
\]

But

\[
\frac{d}{dt} F^2(t) = F(t) \frac{d}{dt} F(t) + \left[ \frac{d}{dt} F(t) \right] F(t), \\
= \int_0^t A(\tau) d\tau A(t) + A(t) \int_0^t A(\tau) d\tau \neq 2A(t) \int_0^t A(\tau) d\tau,
\]

since \( A(t) \) and \( \int_0^t A(\tau) d\tau \) do not necessarily commute!

Therefore

\[
\frac{d}{dt} x(t) = \frac{d}{dt} e^{\int_0^t A(\tau) d\tau} x(0) \neq A(t) e^{\int_0^t A(\tau) d\tau} x(0) = A(t)x(t)
\]

Remember that for LTI \( A(t) = A \) and \( \int_0^t A(\tau) d\tau = At \) commutes with \( A \)!
1.2.1 Equivalent transformations for LTV systems

LTV in state space

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \\
y(t) = C(t)x(t) + D(t)u(t).
\]

Let \(P(t)\) be nonsingular for all \(t\) and define

\[
x(t) = P(t)z(t)
\]

such that

\[
\dot{x}(t) = \dot{P}(t)z(t) + P(t)\dot{z}(t) = A(t)P(t)z(t) + B(t)u(t), \\
\Rightarrow P(t)\dot{z}(t) = [A(t)P(t) - \dot{P}(t)]z(t) + B(t)u(t).
\]

Equivalent LTV system

\[
\dot{z}(t) = P(t)^{-1} [A(t)P(t) - \dot{P}(t)] x(t) + P(t)^{-1} B(t)u(t), \\
y(t) = C(t)P(t)z(t) + D(t)u(t).
\]
1.2.2 Fundamental Matrix

$P(t)$ is called a fundamental matrix when

$$\dot{P}(t) = A(t)P(t), \quad |P(t_0)| \neq 0.$$ 

If $P(t)$ is a fundamental matrix then

$$\dot{z}(t) = P(t)^{-1}B(t)u(t) \Rightarrow z(t) = z(t_0) + \int_{t_0}^{t} P(\tau)^{-1}B(\tau)u(\tau)d\tau.$$ 

and

$$x(t) = P(t)z(t_0) + \int_{t_0}^{t} P(t)P(\tau)^{-1}B(\tau)u(\tau)d\tau,$$

$$= P(t)P(t_0)^{-1}x(t_0) + \int_{t_0}^{t} P(t)P(\tau)^{-1}B(\tau)u(\tau)d\tau,$$

$$= \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau, \quad \Phi(t,\tau) = P(t)P(\tau)^{-1}.$$ 

1.2.3 State Transition Matrix

$\Phi(t,\tau)$ is called the state transition matrix

Properties

1) $\Phi(t,t) = I,$

2) $\Phi^{-1}(t,\tau) = \Phi(\tau,t),$

3) $\Phi(t_1,t_2) = \Phi(t_1,t_0)\Phi(t_0,t_2).$

4) $\frac{d}{dt}\Phi(t,\tau) = A\Phi(t,\tau), \quad \Phi(\tau,\tau) = I.$

Proof:

1) $\Phi(t,t) = P(t)P^{-1}(t) = I,$

2) $\Phi^{-1}(t,\tau) = [P(t)P^{-1}(\tau)]^{-1} = P(\tau)P^{-1}(t) = \Phi(\tau,t),$

3) $\Phi(t_1,t_2) = P(t_1)P^{-1}(t_2)P(t_2)P^{-1}(t_3) = P(t_1)P^{-1}(t_3) = \Phi(t_1,t_3).$

4) $\frac{d}{dt}\Phi(t,\tau) = \frac{d}{dt}P(t)P(\tau)^{-1} = \dot{P}(t)P(\tau) = A(t)P(t)P(\tau) = A(t)\Phi(t,\tau).$
1.2.4 Complete solution

\[ y(t) = C(t)x(t) + D(t)u(t), \]
\[ = C(t)\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t). \]

For SIMO we have

\[ y(t) = C(t)\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} [C(t)\Phi(t, \tau)B(\tau) + D(\tau)\delta(t - \tau)] u(\tau)d\tau, \]
\[ = C(t)\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} h(t, \tau)u(\tau)d\tau, \]

where

\[ h(t, \tau) = C(t)\Phi(t, \tau)B(\tau) + D(\tau)\delta(t - \tau) \]

is the impulse response.
1.2.5 Floquet theory

There exists $\bar{P}(t)$ that transforms the LTV homogeneous system

$$\dot{x}(t) = A(t)x(t)$$

into the equivalent LTI homogeneous system

$$\dot{z}(t) = \bar{A}z(t)$$

We have already seen one case: $\bar{A} = 0$!

In general, for any $\bar{P}(t)$ nonsingular we have

$$\dot{z}(t) = \bar{P}(t)^{-1} \left[ A(t)\bar{P}(t) - \dot{\bar{P}}(t) \right] z(t)$$

Defining

$$\bar{P}(t) = P(t)e^{-\bar{A}t},$$

where $P(t)$ is any fundamental matrix then

$$\bar{P}(t)^{-1} \left[ A(t)\bar{P}(t) - \dot{\bar{P}}(t) \right] = e^{\bar{A}t} P^{-1}(t) \left[ A(t)P(t)e^{-\bar{A}t} + P(t)e^{-\bar{A}t}\bar{A} - \dot{\bar{P}}(t)e^{-\bar{A}t} \right],$$

$$= e^{\bar{A}t} P^{-1}(t) P(t)e^{-\bar{A}t}\bar{A} + e^{\bar{A}t} P^{-1}(t) \left[ A(t)P(t) - \dot{\bar{P}}(t) \right] e^{-\bar{A}t},$$

$$= \bar{A}.$$

When $\bar{A} = 0$ then $\bar{P}(t) = P(t)$!
### 1.3 Example

LTV system

\[ \dot{x} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x \]

is equivalent to equations

\[
\begin{align*}
\dot{x}_1 &= 0, \\
\dot{x}_2 &= tx_1.
\end{align*}
\]

\[
\implies \begin{cases} 
  x_1(t) = x_1(t_0) \\
  x_2(t) = x_2(t_0) + \frac{1}{2}(t^2 - t_0^2)x_1(t_0)
\end{cases}
\]

Fundamental matrix for \( t_0 = 0 \)

\[
\begin{align*}
(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{align*}
\]

\[
\implies \begin{cases} 
  x_1(t) = \begin{bmatrix} 1 \\ \frac{1}{2}t^2 \end{bmatrix} \\
  x_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{cases}
\]

so that

\[
P(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}t^2 & 1 \end{bmatrix}.
\]

State transition matrix

\[
\Phi(t, t_0) = P(t)P^{-1}(t_0),
\]

\[
= \begin{bmatrix} 1 & 0 \\ \frac{1}{2}t^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2}t_0^2 & 1 \end{bmatrix}^{-1},
\]

\[
= \begin{bmatrix} 1 & 0 \\ \frac{1}{2}t^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}t_0^2 & 1 \end{bmatrix},
\]

\[
= \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}(t^2 - t_0^2) & 1 \end{bmatrix}.
\]