Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a handheld calculator with no communication capabilities.

2. You have 170 minutes.

3. Do not forget to write your name and student number.

4. This exam has 6 questions, for a total of 60 points and 2 bonus points.

1. Circuit Equivalence

Consider the circuits in Fig. 1 and answer the following questions:

(a) (3 points) Calculate the equivalent resistance as seen from A and B, A and C, and B and C.

Solution: From A to B, A to C, and B to C, the equivalent resistances are all equal to

\[ R_{AB} = R_{BC} = R_{AC} = R + R = 2R \]  

(+3 points)

(b) (3 points) Calculate the equivalent resistance as seen from D to E, D to F, and E to F.

Solution: From D to E, D to F, and E to F, the equivalent resistances are all equal to S in parallel with 2S, that is

\[ R_{DE} = R_{EF} = R_{DF} = \frac{1}{S + \frac{1}{2S}} = \frac{2S}{3} \]  

(+3 points)
(c) (3 points) Calculate a formula for $S$ as a function of $R$ so that both circuits are completely equivalent.

**Solution:** For the circuits to be equivalent

\[ R_{AB} = R_{BC} = R_{AC} = R_{DE} = R_{EF} = R_{DF} \]  
(+1 point)

that is

\[ 2R = \frac{2}{3}S \]  
(+1 point)

that is

\[ S = 3R \]  
(+1 point)

(d) (1 point) Explain what would change in the formula you obtained in part (c) if the resistances $R$ and $S$ were replaced by impedances in the $s$-domain.

**Solution:** Nothing would change, the formula would remain the same with the resistances replaced by impedances.  
(+1 point)

NOTE TO GRADERS: Some students might bring up potential initial conditions and that is fine.
2. LED Driver

A student needs to drive four LEDs using the digital output of a microcontroller. Each LED is rated at 1.2 V and 30 mA. Because the microcontroller output pin can output a maximum of 20 mA, the student needs to construct the circuit in Fig. 2 to drive the required 4 × 30 mA. Answer the following questions to help him or her select appropriate values for the resistors \( R_B \) and \( R_C \).

(a) (4 points) Assume that the BJT transistor is saturated and calculate the value of \( R_C \) so that each LED has a voltage drop of 1.2 V with a current of 30 mA. Calculate the corresponding value of \( v_2 \) and the power absorbed by the resistor \( R_C \).

\[ R_C = \frac{v_2}{i_c} = \frac{3.8}{0.12} \approx 31.7 \Omega \]

The corresponding power absorbed by the resistor is

\[ P_{R_C} = 3.8 \times 0.12 = 456 \text{mW} \]

(b) (4 points) Let \( v_\gamma = 0.6 \text{ V} \), a transistor current gain \( \beta = 100 \), and calculate the value of \( R_B \) so that when \( v_1 = 5 \text{ V} \) and each LED is operating at their rated voltage and current the transistor is saturated. Calculate the current through \( R_B \) and the power it absorbs.

\[ R_B = \frac{v_1 - v_\gamma}{i_b} = \frac{5 - 0.6}{1.2 \times 10^{-3}} \approx 3.7 \text{K}\Omega \]

The corresponding power absorbed by the resistor is

\[ P_{R_B} = (5 - 0.6) \times 1.2 \times 10^{-3} = 5.28 \text{mW} \]

(c) (2 points) Explain how the circuit you helped design works as \( v_1 \) takes the values \( v_1 = 0 \text{ V} \) and \( v_1 = 5 \text{ V} \). Does it keep the microcontroller output pin current below
Solution:

When \( v_1 = 0 \) V the transistor is cut, and therefore no current flows through the LEDs. They remain off. When \( v_1 = 5 \) V the transistor is saturated and the LEDs are on operating at their rated voltage and current for the values of \( R_B \) and \( R_C \) designed in parts (a) and (b). (+1 point)

The maximum current provided by the microcontroller pin in \( i_b = 1.2 \) mA when \( v_1 = 5\) V, which is well below the rated 20 mA. (+1 point)

3. OpAmp Design

In this question you will design an opAmp circuit to realize the transfer-function:

\[
T(s) = \frac{10(s + 1,000)}{s^2 + 10,010s + 100,000}
\]

(a) (3 points) Calculate positive real numbers \( \alpha \) and \( \beta \) so that

\[
T(s) = \frac{\alpha}{s + 10} + \frac{\beta}{s + 10,000}
\]

Solution:

The above factorization is an expansion of \( T(s) \) in partial-fractions with \( (s + 10) \) and \( (s + 10,000) \) being the factors of the polynomial \( s^2 + 10010s + 100000 = (s + 10)(s + 10000) \). (+1 point)

The values of \( \alpha \) and \( \beta \) are therefore:

\[
\alpha = \lim_{s \to -10} (s + 10)T(s) = \lim_{s \to -10} \frac{10(s + 1000)}{s + 10000} = \frac{990}{999} = \frac{110}{111} \approx 1.0 \\
\beta = \lim_{s \to -10000} (s + 10000)T(s) = \lim_{s \to -10000} \frac{10(s + 1000)}{s + 10} = \frac{9000}{999} = \frac{1000}{111} \approx 9.0
\]

(b) (7 points) Design a circuit using opAmps that implements \( T(s) \) as the sum of the two transfer-functions from part (a).

Solution:

NOTE TO GRADERS: There are many possible solutions to this problem.

NOTE TO GRADERS: Students get no credit if they do not attempt to realize the form worked out in (a).

NOTE TO GRADERS: Students get full credit on this part if they do in terms of \( \alpha \) and \( \beta \) without having calculated them in part (a).

One solution is an inverter summer followed by a gain of \(-1\). (+1 point)
Each term is a low pass filter and one can realize them with an inverter summer as in the following circuit, already in the $s$-domain:

In this circuit, the transfer-function from $V_i$ to $V_o$ is given by

$$\frac{V_1(s)}{V_i(s)} = \frac{R_3}{R_1 + \frac{1}{sC_1}} + \frac{R_3}{R_2 + \frac{1}{sC_2}} = \frac{R_3/R_1}{s + \frac{1}{R_1C_1}} + \frac{R_3/R_2}{s + \frac{1}{R_2C_2}}$$

By setting

$$\frac{R_3}{R_1} = \alpha = 1, \quad \frac{R_3}{R_2} = \beta = 9, \quad \frac{1}{R_2C_2} = 10, \quad \frac{1}{R_1C_1} = 10000$$

which is always, for example, by letting $R = R_3 = 900K$ from which

$$R_1 = 900K\Omega, R_2 = 100K\Omega, \quad C_2 = \frac{1}{10R_2} = 10^{-6}F, C_3 = \frac{1}{10^4R_1} \approx 110 \times 10^{-12}F$$

4. Circuit Analysis

Let $v_x(0) = 1$ V and answer the following questions:

(a) (4 points) Convert the circuit in Fig. 3 to the $s$-domain and formulate its node-
voltage equations. Use the node-voltage and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve node-voltages.

**Solution:**

With an eye on node-voltage analysis we convert to the $s$-domain:

We write the node-voltage equations by inspection omitting node $B$:

$$
\begin{bmatrix}
\frac{2}{R} & -\frac{1}{R} & 0 \\
0 & -\frac{1}{R} - sC & \frac{1}{R} + sC \\
\end{bmatrix}
\begin{bmatrix}
v_A(s) \\
v_B(s) \\
v_C(s) \\
\end{bmatrix}
= \begin{bmatrix}
-I_2(s) \\
I_2(s) + Cv_x(0) \\
\end{bmatrix}
$$

along with the additional relation:

$$V_B(s) = V_1(s)$$

The above equations need to be solved for the node voltages $V_A(s), V_B(s)$ and $V_C(s)$.

(b) (4 points) Convert the circuit in Fig. 3 to the $s$-domain and formulate its mesh-current equations. Use the mesh-currents and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve mesh-currents.

**Solution:**

With an eye on mesh-current analysis we convert to the $s$-domain:
We write the mesh-current equations by inspection omitting mesh $b$:

$$
\begin{bmatrix}
2R & -R & 0 \\
0 & -\frac{1}{sC} & R + \frac{1}{sC}
\end{bmatrix}
\begin{bmatrix}
I_a(s) \\
I_b(s) \\
I_c(s)
\end{bmatrix}
= \begin{bmatrix}
-V_1(s) \\
\frac{v_x(0)}{s}
\end{bmatrix}
$$

along with the additional relation:

$$I_b(s) = I_2(s)$$

The above equations need to be solved for the mesh currents $I_a(s)$, $I_b(s)$ and $I_c(s)$.  

(c) (2 points) Write $s$-domain expressions for the current through the voltage source $v_1$ and the voltage across the current source $i_2$ in terms of the node voltages and mesh currents from parts (a) and (b).

**Solution:**

NOTE TO GRADERS: Students are free to pick their polarities and current directions but they have to be consistent.

The voltage across the source $I_2(s)$, labeled $V_2(s)$, is given in terms of the node voltages by

$$V_2(s) = V_C(s) - V_A(s)$$

or by

$$V_2(s) = \frac{v_x(0)}{s} + \frac{1}{sC}I_b(s) + R(I_b(s) - I_a(s))$$

in terms of the mesh currents.

The current through the source $V_1(s)$, labeled $I_1(s)$, is given in terms of the mesh currents by

$$I_1(s) = -I_a(s)$$
or by

\[ I_1(s) = \frac{V_B(s) - V_A(s)}{R} + \left( sC + \frac{1}{R} \right) (V_B(s) - V_C(s)) + Cv_x(0) \]  

(+1/2 point)

in terms of the node voltages.
5. **OpAmp Circuit Analysis**

Assume zero initial-conditions for the circuits in Fig. 4 and answer the questions:

(a) (7 points) Convert the three circuits into the $s$-domain and show that they have identical transfer-functions

\[
\frac{V_o(s)}{V_i(s)} = \frac{-10s}{s + \frac{1}{RC}}.
\]

**Solution:**

Converting into the $s$-domain under zero-initial conditions.
These are all based on the inverter opAmp circuits. The transfer-function for the first circuit is given by a gain followed by a voltage divider

\[ Z_1(s) = 10R, \quad Z_2(s) = R, \quad K(s) = \frac{R}{R + 1/(RC)} = \frac{s}{s + 1/(RC)} \]

\[ T(s) = -K(s) \frac{Z_1(s)}{Z_2(s)} = -\frac{10s}{s + 1/(RC)} \]  

(+) points

The transfer-function for the second circuit is given by

\[ Z_1(s) = 10R, \quad Z_2(s) = R + \frac{1}{sC} = \frac{R(s + 1/(RC))}{s} \]

\[ T(s) = -\frac{Z_1(s)}{Z_2(s)} = -\frac{10Rs}{R(s + 1/(RC))} = -\frac{-10s}{s + 1/(RC)} \]  

(+) points

The transfer-function for the third circuit is given by

\[ Z_1(s) = \frac{1}{10R + \frac{sC}{10}} = \frac{10/C}{s + 1/(RC)}, \quad Z_2(s) = \frac{1}{sC}, \]

\[ T(s) = -\frac{Z_1(s)}{Z_2(s)} = -\frac{10/C}{s + 1/(RC)} = -\frac{-10s}{s + 1/(RC)} \]  

(+) points

They are all identical, as stated.

(b) (3 points) Let \( v_i(t) = V_s u(t) \) and calculate the response \( v_o(t) \).

Solution:

Under zero-initial conditions, the response of all circuits is identical and is given by

\[ V_o(s) = \frac{-10s}{s + \frac{1}{RC}} V_i(s) = \frac{-10s}{s + \frac{1}{RC}} \frac{V_s}{s + \frac{1}{RC}} = \frac{-10V_s}{s + \frac{1}{RC}} \]  

(+) points

Using the Laplace inverse:

\[ v_o(t) = -10V_s L^{-1} \left\{ \frac{1}{s + \frac{1}{RC}} \right\} = -10V_s e^{-\frac{t}{RC}} u(t) \]  

(+) points

6. Frequency Response

Consider the circuits in Fig. 4 and answer the following questions:

(a) (3 points) Calculate the magnitude and phase of the circuits’ frequency response at \( \omega = 0, \omega = (RC)^{-1}, \) and \( \omega \to \infty \).
Solution:
The frequency-response function is

\[ T(j\omega) = \frac{-10j\omega}{j\omega + \frac{1}{RC}} \]  

(+1/2 point)

from which

\[ |T(j\omega)| = \frac{10|\omega|}{\sqrt{\omega^2 + \frac{1}{(RC)^2}}} \]  

(+1/2 point)

and

\[ \angle T(j\omega) = -\pi/2 - \tan^{-1}\omega RC, \quad \omega > 0 \]  

(+1/2 point)

Calculating at \( \omega = 0 \)

\[ |T(j0)| = 0 \quad \angle T(j0) = -\pi/2 \]  

(+1/2 point)

At \( \omega = (RC)^{-1} \)

\[ |T(j(RC)^{-1})| = \frac{10}{\sqrt{2}} = 5\sqrt{2} \quad \angle T(j(RC)^{-1}) = -3\pi/4 \]  

(+1/2 point)

At \( \omega \to \infty \)

\[ |T(j\infty)| = 10 \quad \angle T(j\infty) = -\pi \]  

(+1/2 point)

(b) (4 points) Sketch the magnitude and phase of the circuits’ frequency response. What kind of filter is the circuit?

Solution:
A sketch of the magnitude is as in the following plot:

A sketch of the phase is as in the following plot:
This circuit is a high-pass filter. (+1 point)

(c) (3 points) Calculate the steady-state response $v_{o}^{SS}(t)$ produced in response to an input voltage $v_i(t) = 1 + \cos((RC)^{-1}t)$.

**Solution:**
The steady-state response can be calculated using the frequency response method and linearity. (+1 point)
That is

$$v_{o}^{SS}(t) = |T(j0)| \cos(\angle T(j0)) + |T(j(RC)^{-1})| \cos((RC)^{-1}t + \angle T(j(RC)^{-1}))$$
$$= 5\sqrt{2} \cos((RC)^{-1}t - \frac{3\pi}{4})$$  (+2 point)

(d) (2 points (bonus)) Which of the circuits has a problem with saturation? Explain.

**Solution:** The output of the first circuit might become saturated if the signal $v_i$ has a non-zero DC component, such as in part (c). The other two circuits do not suffer from this problem.  (+2 bonus points)

Have a great Winter break!