Problems

P3.1. Compute the Laplace transform of the following signals \( f(t) \) defined for \( t \geq 0 \):

a) \( 1 - e^{-t} \);

b) \( e^{-t} - e^{-2t} \);

c) \( \sinh(t) - \cosh(2t) \);

d) \( \sin(t) - \cos(t) \);

e) \( t + te^{-t} \);

f) \( \sin(t) - e^{-t} \cos(2t) \);

g) \( e^{-t} \cos(t + \pi/4) \);

h) \( 1(t) + 1(t - 1) \);

i) \( e^{-t} + 1(t - 1)e^{-t-1} \);

j) \( \delta(t) + t e^{-t} \sin(t) \);

P3.2. Compute the inverse Laplace transform of the following complex valued functions \( F(s) \):

a) \( \frac{1}{s(s + 1)} \);

b) \( \frac{s - 1}{s + 1} \);

c) \( \frac{s - 1}{s(s + 1)} \);

d) \( \frac{s}{s^2 + 2s + 1} \);

e) \( \frac{1}{s^2 - 1} \);

f) \( \frac{1}{s^2(s + 2)^2} \);

g) \( \frac{1}{(s + 1)^2 + 1} \);

h) \( \frac{1 - e^{-s}}{s} \);

i) \( \frac{s + 1 - e^{-s}}{s(s + 1)} \);

j) \( \frac{1}{s + 1} - \frac{s}{(s + 1)^2} + \frac{s^2}{(s + 1)^3} \);

P3.3. Prove that when \( f(t) \geq 0 \), is such that \( f(t) = 0 \), all \( t \geq T \) then its Laplace transform is analytic in \( \mathbb{C} \) (entire) or its singularities are removable. Verify this by showing that the following complex valued functions:

a) \( 1 - e^{-s} + e^{-2s} \);

b) \( \frac{1 - e^{-s}}{s} \);

c) \( \frac{1 - 2e^{-s} + e^{-2s}}{s} \);

d) \( \frac{1 - 2e^{-s} + e^{-2s}}{s^2} \);

e) \( \frac{1 - 2e^{-s} + 2e^{-3s} - e^{-4s}}{s^2} \);

f) \( \frac{1 - 2e^{-s} + 2e^{-3s} - e^{-4s}}{s^3} \);

g) \( \frac{e^{-s-1}(e^{-s} - e)}{s - 1} \);

P3.4. The complex valued functions \( F(s) \):

a) \( \frac{e^{-s}}{s} \);

b) \( \frac{1 - e^{-2\pi s}}{s^2 + 1} \);

c) \( \frac{s(1 - e^{-2\pi s})}{s^2 + 1} \);

are Laplace transforms obtained from functions \( f(t) \), \( t \geq 0 \). Assume that \( f(0^-) = 0 \) and use the formal derivative property and the inverse Laplace transform to calculate the time-derivative function, \( f'(t) \), \( t \geq 0 \), without evaluating \( f(t) \). Locate the discontinuities of the original function \( f(t) \).

P3.5. Show that

\[ \mathcal{L} \left\{ \frac{\sin(at)}{t} \right\} = \tan^{-1} \left\{ \frac{a}{s} \right\} . \]

Hint: Use \( \int_0^a \cos(\tau) d\tau = \sin(\tau)/t \) and the linearity property.

P3.6. Use P3.5, the (t-domain) integration property and the final value property to show that

\[ \lim_{t \to \infty} \int_{0}^{t} \frac{\sin(at)}{t} dt = \operatorname{sign}(a) \frac{\pi}{2} . \]

P3.7. Use P3.6 and (3.7) to show that

\[ \int_{-\infty}^{\infty} \delta(t) dt = 1. \]

P3.8. Use the convolution property to show that

\[ \int_{0}^{t} f(\tau) \delta(t - \tau) d\tau = f(t) . \]
P3.9. Let $C$ be the unit circle traversed in the counter-clockwise direction. Use Theorem 3.1, Cauchy’s Residue Theorem, to evaluate the contour integral:

$$\int_C f(s) \, ds$$

for the following complex valued functions:

a) $1 - e^{-s}$;

b) $\frac{1}{s} + e^{-s}$;

c) $\frac{1}{s}$;

d) $\frac{1}{s^2}$;

e) $\frac{1}{s(s + 1/2)}$;

f) $\frac{s}{(s + 1/2)(s + 2)}$;

P3.10. Compute the expansion in partial fractions of the following rational functions:

a) $\frac{1}{(s + 1)(s + 2)}$;

b) $\frac{s - 1}{(s + 1)(s + 2)}$;

c) $\frac{1}{s(s + 1)(s + 2)}$;

d) $\frac{1}{s(s + 1)(s + 2)}$;

e) $\frac{1}{s^2(s + 1)}$;

f) $\frac{1}{s^2 + 2s + 1}$;

g) $\frac{1}{s^2 + 2s + 2}$;

h) $\frac{s + 1}{(s^2 + 2s + 2)^2}$;

P3.11. Let $u(t)$, $t \geq 0$, be such that $u(t) = 0$ for all $t \geq T$ and $U(s)$ be its Laplace transform. Show that

$$y(t) = \mathcal{L}^{-1}\{U(s)\}, \quad Y(s) = \frac{U(s)}{1 - e^{-sT}}$$

is a periodic function with period $T$. Use this fact to calculate and sketch the plot of the inverse Laplace transform of

$$Y(t) = \frac{1 - e^{-sT/2}}{s(1 - e^{-sT})}.$$
P3.15. Show that the response of sample-and-hold system, P3.14, to an impulse \( \delta(t - \tau) \) is
\[
g(t, \tau) = \delta(kT - \tau),
\]
\[
kT \leq t < (k + 1)T, \quad k \in \mathbb{N}.
\]
and that
\[
y(t) = \int_0^t g(t, \tau) u(\tau) \, d\tau.
\]

Hint: Use P3.7.

In the following questions, when asked to compute the response to a given input, assume zero initial conditions unless otherwise noted.

P3.16. A linear time-invariant system has as impulse response
\[
g(t) = e^{-t}, \quad t \geq 0.
\]
Compute the system’s transfer-function. What is the order of the system? Is the system asymptotically stable? Calculate the response to a constant input \( u(t) = \bar{u}, \quad t \geq 0 \). Identify the transient and steady-state part of the response.

P3.17. Repeat P3.16 for the following impulse responses, \( g(t), \quad t \geq 0 \):

a) \( e^{-2t} - e^{-t} \);

b) \( e^t - e^{-t} \);

c) \( -te^{-2t} \);

d) \( e^{-t} \cos(t) \);

e) \( e^{-t} (\cos(t) - \sin(t)) \);

f) \( \cos(t + \pi/6) \);

g) \( \delta(t) + t e^{-t} \sin(t) \);

h) \( 1(t) - 1(t - 1) \);

i) \( t - 2(t - 1)(t - 1) + (t - 2)(t - 2) \);

P3.18. You have shown in P2.1 that the ordinary differential equation
\[
m \ddot{v} + b v = mg
\]
is a simplified description of the motion of an object of mass \( m \) dropping vertically under constant gravitational acceleration, \( g \), and linear air resistance, \(-b v\). Solve this differential equation using the Laplace transform. Treating the gravitational force, \( u = mg \), as an input, calculate the transfer-function from the input, \( u \), to the velocity, \( v \), then calculate the transfer-function from the input, \( u \), to the object’s position, \( x = \int_0^t v(\tau) \, d\tau \). Assume that all constants are positive. Are these transfer-functions asymptotically stable?

P3.19. You have shown in P2.7 that the ordinary differential equation
\[
(J_1 \dot{\theta}_1^2 + J_2 \dot{\theta}_2^2) \dot{\omega}_1 = r_2^2 \tau,
\]
\[
\dot{\omega}_2 = (r_1/r_2) \omega_1
\]
is a simplified description of the motion of a rotating machine driven by a belt without slip as in Fig. 2.18, where \( \omega_1 \) is the angular velocity of the driving shaft and \( \omega_2 \) is the machine’s angular velocity. Calculate the transfer-function from the input torque, \( \tau \), to the machine’s angular velocity, \( \omega_2 \), then calculate the transfer-function from the torque, \( \tau \), to the machine’s angular position, \( \theta_2 = \int_0^t \omega_2(\tau) \, d\tau \). Assume that all constants are positive. Are these transfer-functions asymptotically stable?

P3.20. Consider the simplified model of the rotating machine described in P3.19. Calculate the machine’s angular position, \( \theta_2 \), and angular velocity, \( \omega_2 \), obtained in response to a constant torque input, \( \tau(t) = \bar{\tau}, \quad t \geq 0 \). Identify the transient and the steady-state component of the responses. Repeat the question for an input \( \tau(t) = \bar{\tau} \cos(\omega t), \quad t \geq 0 \).

P3.21. Recalculate the steady-state responses to the inputs in P3.20 using the system’s frequency response. Compare your answers.

P3.22. You have shown in P2.7 that the ordinary differential equation
\[
J \ddot{\omega} + (b_1 + b_2) \omega = \tau + g r (m_1 - m_2),
\]
\[
J = J_1 + J_2 + r^2 (m_1 + m_2),
\]
\[
v_1 = r \omega,
\]
is a simplified description of the motion of the elevator in Fig. 2.19, where \( \omega \) is the angular velocity of the driving shaft and \( v_1 \) is the elevator’s load linear velocity. Treating the gravitational torque, \( \tau = g r (m_1 - m_2) \), as an input, calculate the transfer-function, \( G_w \), from the gravitational torque, \( \tau \), to the elevator’s load linear velocity, \( v_1 \), assuming that the motor torque, \( \tau \), is zero. Calculate the transfer-function, \( G_\tau \), from the motor torque, \( \tau \), to the elevator’s load linear velocity, \( v_1 \), assuming that the gravitational torque, \( \tau \), is zero. Assume that all constants are positive. Are these transfer-functions asymptotically stable? Assume zero initial conditions and verify that:
\[
V_1(s) = G_\tau(s) \tau(s) + G_w(s) W(s).
\]

Explain the results.
Consider the simplified model of the elevator described in P3.22. Calculate the elevator’s linear velocity, \( v_1 \), obtained in response to a constant torque input, \( \tau(t) = \tau, t \geq 0 \). Identify the transient and the steady-state component of the response. Repeat the question for an input \( \tau(t) = \tau \cos(\omega t), t \geq 0 \).

P3.24. Recalculate the steady-state responses to the inputs in P3.23 using the system’s frequency response. Compare your answers.

P3.25. You have shown in P2.16 that the ordinary differential equation

\[
J \dot{\omega} + \left( b + \frac{K_t K_e}{R_a} \right) \omega = \frac{K_t}{R_a} v_a
\]

is a simplified description of the motion of the rotor of the DC motor in Fig. 2.20, where \( \omega \) is the rotor angular velocity. Calculate the transfer-function from the armature voltage, \( v_a \), to the rotor’s angular velocity, \( \omega \). Assume that all constants are positive. Is this transfer-function asymptotically stable?

P3.26. Consider the simplified model of the DC motor described in P3.25. Calculate the motor’s angular velocity, \( \omega \), obtained in response to a constant armature voltage input, \( v_a(t) = \bar{v}_a, t \geq 0 \). Identify the transient and the steady-state component of the response. Repeat the question for \( v_a(t) = \bar{v}_a \cos(\omega t), t \geq 0 \).

P3.27. Recalculate the steady-state responses to the inputs in P3.26 using the system’s frequency response. Compare your answers.