Problems

P4.1. Calculate

\[ S = \frac{1}{1 + GK}, \quad D = \frac{G}{1 + GK} \]
\[ H = \frac{GK}{1 + GK}, \quad Q = \frac{K}{1 + GK} \]

for the system and controller transfer-functions:

\[ G(s) = \frac{a}{s + a}, \quad K(s) = K. \]

What is the order of each transfer-function?

P4.2. Repeat P4.1 for the following system and controller transfer-functions:

a) \( G = \frac{a}{s + a}, \quad K = \frac{K_p}{s}; \)

b) \( G = \frac{a}{s + a}, \quad K = \frac{K_p + K_i}{s}; \)

c) \( G = \frac{a}{s + a}, \quad K = \frac{K_p s + c}{s + b}; \)

d) \( G = \frac{a_2}{s^2 + a_1 s + a_2}, \quad K = K_p; \)

e) \( G = \frac{a_2}{s^2 + a_1 s + a_2}, \quad K = K_p + K_d s; \)

f) \( G = \frac{a_2}{s^2 + a_1 s + a_2}, \quad K = K_p + \frac{K_i}{s}. \)

P4.3. Consider the standard feedback connection in Fig. 4.16. Is the closed-loop system asymptotically stable if the system and controller transfer-functions are:

\[ G = \frac{1}{s}, \quad K = 1, \]

Does the closed-loop system achieve asymptotic tracking of a constant input \( y(t) = \tilde{y}, t \geq 0? \)

P4.4. Repeat P4.3 for the following combinations of system and controller transfer-functions:

a) \( G = \frac{1}{s}, \quad K = \frac{1}{s + 1}; \)

b) \( G = \frac{1}{s}, \quad K = \frac{1}{s}; \)

c) \( G = \frac{1}{s + 1}, \quad K = 1; \)

P4.5. Consider the closed-loop connection in Fig. 4.17 with \( v = 0 \) and the combinations of system and controller transfer-functions in P4.3 and P4.4. Does the closed-loop system achieve asymptotic rejection of a constant disturbance \( w(t) = \tilde{w}, t \geq 0? \)

P4.6. Consider the closed-loop connection in Fig. 4.17 with \( v = 0 \) and the system and controller transfer-functions:

\[ G = \frac{1}{s}, \quad K = \frac{(s + 1)^2}{s^2 + 1}. \]

Show that the closed-loop system asymptotically tracks a constant reference input \( y(t) = \tilde{y}, t \geq 0 \), and asymptotically rejects an input disturbance \( w(t) = \tilde{w} \cos(\omega t), t \geq 0 \).

P4.7. Consider the closed-loop connection in Fig. 4.17 with \( w = 0 \) and the system and controller transfer-functions:

\[ G = \frac{1}{s + 1}, \quad K = \frac{1}{s}. \]

Calculate the steady-state component of the output, \( y \), and the tracking error, \( e = \tilde{y} - y \), in response to:

\( \tilde{y}(t) = \tilde{y}, \quad v(t) = \tilde{v} + \cos(\omega t), t \geq 0 \).

Does the closed-loop achieve asymptotic tracking of the reference input, \( \tilde{y}(t) \)? Does the closed-loop achieve asymptotic rejection of the measurement noise input, \( v(t) \)? What happens if \( \omega \) is very large and \( \tilde{v} \) is zero?
P4.8. Repeat P4.1 for the following system and controller transfer-functions:

a) \[ G = \frac{1}{s + 1}, \quad K = \frac{s + 1}{s + 2}; \]

b) \[ G = \frac{1}{s - 1}, \quad K = \frac{s - 1}{s + 1}; \]

c) \[ G = \frac{1}{s(s + 1)}, \quad K = \frac{s}{s + 1}; \]

d) \[ G = \frac{s - 1}{s + 1}, \quad K = \frac{1}{s - 1}; \]

Is the closed-loop system internally stable? Explain your reasoning.

P4.9. The controller in Fig. 4.18 is known as a two-degrees-of-freedom controller. The transfer-function \( K \) is the feedback part of the controller and the transfer-function \( F \) is the feedforward part of the controller. Show that

\[ y = (H + FD)\ddot{y}, \]

where

\[ H = \frac{GK}{1 + GK}, \quad D = \frac{G}{1 + GK}. \]

How does the choice of the feedforward term \( F \) affect closed-loop stability? Name one advantage and one disadvantage of this scheme if you are free to pick any suitable feedforward, \( F \), and feedback, \( K \), transfer-functions.

P4.10. With respect to P4.9 and the block diagram in Fig. 4.18 show that if

\[ F = G^{-1} \]

then \( y = \ddot{y} \) regardless of \( K \). Compare this with the open-loop solution discussed in § 1. Is it possible to chose \( F = G^{-1} \) when: a) \( G \) is not asymptotically stable? b) \( G \) has a zero on the right-hand side of the complex plane? c) \( G \) has a delay in the form \( G(s) = e^{-\tau s} \bar{G}(s), \tau > 0? \)

P4.11. Let

\[ G(s) = \frac{s + 1}{s - 1}, \quad K(s) = K. \]

Select \( K \) and design \( F \) in the block diagram in Fig. 4.18 so that \( y = \ddot{y} \). If \( \ddot{y} \) is a unit step, what is the corresponding signal \( w \)?

P4.12. You have shown in P2.7 that the ordinary differential equation

\[ (J_1 \dot{r}_1^2 + J_2 \dot{r}_2^2) \omega_1 = r_2^2 \tau, \quad \omega_2 = (r_1/r_2) \omega_1 \]

is a simplified description of the motion of a rotating machine driven by a belt without slip as in Fig. 2.18, where \( \omega_1 \) is the angular velocity of the driving shaft and \( \omega_2 \) is the machine’s angular velocity. Let \( r_1 = 0.05 \text{m}, r_2 = 0.25 \text{m}, \]

\( m_1 = 1 \text{kg}, m_2 = 10 \text{kg}, b_1 = 0.125 \text{kg m/s}, \]

\( b_2 = 6.25 \text{kg m/s}, J_i = m_i r_i^2 / 2, i = 1, 2 \).

Design a feedback controller:

\[ \tau = K(\theta_2 - \theta_2), \quad \theta_2 = \int_0^t \omega_2(\tau) \, d\tau, \]

and select \( K \) such that the closed-loop system is internally stable. Can the closed-loop system asymptotically track a constant angular reference \( \theta_2(t) = \theta_2, t \geq 0? \) Explain.

P4.13. You have shown in P2.10 that the ordinary differential equation

\[ J \ddot{\omega} + (b_1 + b_2) \omega = \tau + g r(m_1 - m_2), \]

\[ J = J_1 + J_2 + r^2(m_1 + m_2), \]

\( v_1 = r \omega \),

is a simplified description of the motion of the elevator in Fig. 2.19, where \( \omega \) is the angular velocity of the driving shaft and \( v_1 \) is the elevator’s load linear velocity. Let \( r = 1 \text{m}, m_1 = m_2 = 1000 \text{kg}, b_1 = b_2 = 120 \text{kg m/s}, J_1 = J_2 = 200 \text{kg m}^2, \text{and} \ g = 10 \text{m/s}^2. \)

Design a feedback controller:

\[ \tau = K(\ddot{x}_1 - x_1), \quad x_1 = \int_0^t v_1(\tau) \, d\tau, \]

and select \( K \) such that the closed-loop system is internally stable. Can the closed-loop system asymptotically track a constant position reference \( x_1(t) = \ddot{x}_1, t \geq 0? \) Explain.

P4.14. Repeat P4.13 with \( m_2 = 800 \text{kg}. \)

P4.15. Design a feedback controller so that the closed-loop elevator system in P4.13 can asymptotically track a constant reference \( \ddot{x}_1(t) = \ddot{x}_1, t \geq 0 \) when \( m_2 = 800 \text{kg}. \)