Problems

P5.1. Write state-space equations for the block diagrams in Fig. 5.17 and in Fig. 5.18 and compute the transfer-function from $u$ to $y$.

P5.2. You have shown in P2.1 that the ordinary differential equation

$$m \dot{v} + b v = m g$$

is a simplified description of the motion of an object of mass $m$ dropping vertically under constant gravitational acceleration, $g$, and linear air resistance, $-b v$. Let the gravitational force, $mg$, be the input and the vertical velocity, $v$, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.3. Repeat P5.2 considering the vertical position, $x = \int_0^r v(\tau) d\tau$, as the output.

P5.4. Repeat P5.2 considering the vertical acceleration, $\ddot{v}$, as the output. Hint: Use the original equation to obtain $\ddot{v}$ as a function of $v$.

P5.5. The ordinary differential equation

$$m \dot{v} + b v|v| = m g$$

is a simplified description of the motion of an object of mass $m$ dropping vertically under constant gravitational acceleration, $g$, and quadratic air resistance, $-b v|v|$. Let the gravitational force, $mg$, be the input and the vertical velocity, $v$, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.6. Use the block-diagram obtained in P5.5 to simulate the velocity of an object with $m = 70kg$, $b = 0.0175kg/m$, and $g = 10m/s^2$ falling with zero initial velocity. Compare your solution with the one given in P2.5.

P5.7. Use P5.5 to redo P2.4.

P5.8. Calculate the equilibrium points of the state-space representation obtained in P5.5. Linearize the state-space equations about the equilibrium points and compute the corresponding transfer-functions. Are the equilibrium points asymptotically stable?

P5.9. You have shown in P2.7 that the ordinary differential equation

$$(J_1 r_1^2 + J_2 r_2^2) \ddot{\omega}_1 = r_2^2 \tau, \quad \omega_2 = (r_1/r_2) \omega_1$$

is a simplified description of the motion of a rotating machine driven by a belt without slip as in Fig. 2.18, where $\omega_1$ is the angular velocity of the driving shaft and $\omega_2$ is the machine’s angular velocity. Let the torque, $\tau$, be the input and the machine’s angular velocity, $\omega_2$, be the output.
and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.10. Repeat P5.9 considering the machine’s angle, $\theta_2 = \int_0^t \omega_2(\tau) \, d\tau$, as the output.

P5.11. Repeat P5.9 considering the machine’s angular acceleration, $\dot{\omega}_2$, as the output. Hint: Use the original equation to obtain $\dot{\omega}_2$ as a function of $\omega_1$.

P5.12. You have shown in P2.10 that the ordinary differential equation

$$J \ddot{\omega} + (b_1 + b_2) \omega = \tau + g r (m_1 - m_2),$$

$$J = J_1 + J_2 + r^2 (m_1 + m_2),$$

$$v_1 = r \omega,$$

is a simplified description of the motion of the elevator in Fig. 2.19, where $\omega$ is the angular velocity of the driving shaft and $v_1$ is the elevator’s load linear velocity. Let the torque, $\tau$, and the gravitational torque, $g r (m_1 - m_2)$, be inputs and the elevator’s linear velocity, $v_1$, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.13. Compute the transfer-function from the two inputs to the single output in P5.12. The transfer-function will be a $1 \times 2$ matrix. Hint: Use the state-space equations and equation (5.6).

P5.14. You have shown in P2.16 that the ordinary differential equation

$$J \ddot{\omega} + \left( b + \frac{K_i K_e}{R_a} \right) \omega = \frac{K_i}{R_a} v_a,$$

is a simplified description of the motion of the rotor of the DC motor in Fig. 2.20, where $\omega$ is the rotor angular velocity. Let the armature voltage, $v_a$, be the input and the angular velocity, $\omega$, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.15. Repeat P5.14 considering the rotor’s angle, $\theta_2 = \int_0^t \omega_2(\tau) \, d\tau$, as the output.

P5.16. Recall from P2.16 that the rotor torque:

$$\tau = K_i i_a.$$

The armature current, $i_a$, is related to the armature voltage, $v_a$, and the rotor angular velocity, $\omega$, through

$$v_a = R_a i_a + K_e \omega.$$

Repeat P5.14 considering the rotor’s torque as the output. At what (constant) angular velocity the motor attains its highest torque?

P5.17. You have shown in P2.31 that the ordinary differential equation

$$RC_2 \dot{v}_o + RC_1 \dot{v}_1 + v_1 = 0.$$

is an approximate model for the electric circuit in Fig. 2.27. Let the input voltage, $v_1$, be the input and the output voltage, $v_o$, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.18. You have shown in P2.28 that the ordinary differential equations:

$$m_1 \ddot{x}_1 + (b_1 + b_2) \ddot{x}_2 = (k_1 \ddot{x}_1 - b_2 \ddot{x}_2 - k_2 x_2 = 0,$$

$$m_2 \ddot{x}_2 + b_2 (\ddot{x}_2 - \ddot{x}_1) + k_2 (x_2 - x_1) = f_2$$

constitute a simplified description of the motion of the mass-spring-damper system in Fig. 2.25 where $x_1$ and $x_2$ are displacements and $f_2$ is a force applied on the mass $m_2$. Let the force, $f_2$, be the input and the displacement, $x_2$, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equations in state-space form.

P5.19. Let $m_1 = m_2 = 1$ kg, $b_1 = b_2 = 0.1$ kg/s, $k_1 = 1$ N/m, $k_2 = 2$ N/m. Use MATLAB to compute the transfer-function from the force $f_2$ to the displacement $x_2$. Is this system asymptotically stable?

P5.20. The equations of motion of a rigid-body with principal moments of inertia $J_1$, $J_2$ and $J_3$ is given by Euler’s equations:

$$J_1 \ddot{\omega}_1 + \omega_2 \omega_3 (J_3 - J_2) = 0,$$

$$J_2 \ddot{\omega}_2 + \omega_3 \omega_1 (J_1 - J_3) = 0,$$

$$J_3 \ddot{\omega}_3 + \omega_1 \omega_2 (J_2 - J_1) = 0,$$

where the angular velocity vector

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

is the state vector. Verify that

$$\bar{\omega} = (\omega_1, \omega_2, \omega_3) = (\Omega, 0, 0)$$

is an equilibrium point. Linearize the equations about this equilibrium point. Show that if $J_2 < J_1 < J_3$ or $J_3 < J_1 < J_2$ then this is an unstable equilibrium point. Interpret this result.
P5.21. Compute the linearized equations for the pendulum in a cart model developed in § 5.5. Let\( m_p = 2\, \text{kg}, \, m_c = 10\, \text{kg}, \, \ell = 1\, \text{m}, \, b_p = 0.01\, \text{kg m}^2/\text{s}, \, b_c = 0.1\, \text{km/s}, \, r = \ell/2, \, J_p = m\ell^2/12.\) and use MATLAB to compute the transfer-function from \( u \) to \( \theta \) and from \( u \) to \( \dot{x}_c \) around the equilibrium points calculated with \( \bar{\theta} = 0 \) and \( \bar{\theta} = \pi \) and \( \bar{u} = 0. \) Are the equilibria asymptotically stable?

P5.22. We have shown in § 2.8 that the water level, \( h, \) in a rectangular water tank of cross-section area \( A \) can be modeled as the integrator:

\[
\dot{h} = \frac{1}{A} \dot{w}_{\text{in}}
\]

where \( w_{\text{in}} \) is the in-flow rate. If water is allowed to flow out from the bottom of the tank through an orifice then

\[
\dot{h} = \frac{1}{A} (w_{\text{in}} - w_{\text{out}}).
\]

The out-flow rate can be approximated by

\[
w_{\text{out}} = \frac{1}{R} (p_t - p_a)^{1/\alpha}
\]

where the resistance \( R > 0 \) and the exponent \( \alpha \) depend on the shape of the out-flow orifice, \( p_a \) is the ambient pressure outside the tank, and

\[
p_t = p_a + \rho g h,
\]

is the pressure at the water level, \( \rho \) is the water density and \( g \) is the gravitational acceleration. Combine these equations to write a nonlinear differential equation in state-space relating the water in-flow rate \( w_{\text{in}} \) with the water tank level, \( h. \) Determine a water in-flow rate \( w_{\text{in}} \) such that the tank system is in equilibrium with a water level \( h = \bar{h} > 0. \) Linearize the state-space equations about this equilibrium point for \( \alpha = 2 \) and compute the corresponding transfer-function. Is this equilibrium point asymptotically stable?