P6.14

b) With this system, the root locus simply starts at the pole and ends at the zero. Sketches by hand and \texttt{matlab} are in Figure 1. In \texttt{matlab}, use \texttt{zpk} to build the system with desired poles and zeros and gain, then use \texttt{rlocus} to plot its root locus. For example,
\[
\texttt{sys} = \texttt{zpk([1],[2],1);}
\]
\[
\texttt{rlocus(sys)}
\]

![Root Locus](image)

Figure 1: Root locus for P6.14(b): by hand (left) and \texttt{matlab} (right).

d) The portion from -2 to 0 of the real axis belongs to the root locus. Since we have 3 poles and 1 zeros, there are 2 asymptotes. The intersection of the asymptotes is
\[
c = \frac{(j-j-2) - 0}{3-1} = -1,
\]
and the angles that they make with the real axis are
\[
\phi_1 = \frac{\pi}{2} \quad \text{and} \quad \phi_2 = \frac{\pi + 2\pi}{2} = \frac{3\pi}{2}.
\]

Root locus sketches by hand and \texttt{matlab} are in Figure 2.
The portion from -1 to 1 of the real axis belongs to the root locus. There are no zeros and 4 poles, which means there are 4 asymptotes. Their intersection is
\[
c = \frac{(1 - 1 + j - j) - 0}{4 - 0} = 0,
\]
and the angles they make with the real axis are
\[
\phi_1 = \frac{\pi}{4}, \phi_2 = \frac{3\pi}{4}, \phi_3 = \frac{5\pi}{4}, \phi_4 = \frac{7\pi}{4}.
\]
Root locus sketches by hand and \texttt{matlab} are in Figure 3.
b) Using *sisotool* in *matlab*, one possible controller is

\[ C(s) = 1.8 \times \frac{s + 6}{s - 5} \]

which produces the root locus in Figure 4. Note that the gain can be chosen so that the closed-loop system is stable. The controller is not asymptotically stable because it has a pole at 5.

![Figure 4: Root locus for P6.15(b).](image)

\[
\begin{bmatrix}
  \dot{w}_1 \\
  \dot{w}_2 \\
  \dot{w}_3
\end{bmatrix} = \begin{bmatrix}
  r \\
  \dot{r} \\
  \omega
\end{bmatrix},
\]

\text{P6.15}

\text{d)}

Referring to Part (d) of P6.14, we can see that the system is already stable. Therefore any proportional control of gain \( K > 0 \) will suffice.

\text{e)}

A possible controller has a gain of 0.8 and complex zeros at 0.6 \( \pm j1.3 \) and a real zero at \(-1\) so that a pole/zero cancelation happens, and complex poles at \(-6.5 \pm j2\) and a real pole at \(-19\). The resulting root locus is in Figure 5. Again, the gain can be select differently, making sure that the closed-loop poles are in the left half plane.

\text{P6.21}

Let the states
Figure 5: Root locus for P6.15(e).

and the output $y = r = w_1$. Rearrange the differential equations to get the following state-space form.

$$
\dot{w}_1 = w_2 = f_1(w, u)
$$
$$
\dot{w}_2 = w_1\omega^2 - \frac{GM}{w_1} = f_2(w, u)
$$
$$
\dot{w}_3 = \frac{w}{m} - \frac{2w_2\omega}{w_1} = f_3(w, u)
$$
$$
y = w_1 = h(w, u)
$$

The block diagram is in Figure 6.

Figure 6: Block diagram for P6.21.
To find the equilibrium points, from the state-space found in P6.21, set \( \dot{w} = 0 \).

\[
\begin{align*}
\dot{w}_1 &= 0 \Rightarrow \dot{r} = 0 \\
\dot{w}_2 &= 0 \Rightarrow \dot{w}_1 \omega^2 - \frac{GM}{w_1^2} = 0: \text{ true if } w_1 = r = R, \omega = \Omega \text{ and } \Omega^2 R^3 = GM \\
\dot{w}_3 &= 0 \Rightarrow u = 0
\end{align*}
\]

And so, if \( w_1 = r = R, \omega = \Omega \) and \( \Omega^2 R^3 = GM \), then \( u = \dot{r} = 0, r = R, \omega = \Omega \) is the equilibrium point of the non-linear system.

Define the new states as follows.

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
r - R \\
\dot{r} \\
R \omega - R \Omega
\end{bmatrix},
\]

and the new output \( \hat{y} = r - R \).

Then the equilibrium above becomes

\[
\begin{bmatrix}
u \\
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Linearization of the system around the above equilibrium point is

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 \\
3 \Omega^2 & 0 & 2 \Omega \\
0 & -2 \Omega & 0
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
\frac{1}{m}
\end{bmatrix} u = Ax + Bu
\]

\[
\hat{y} = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix} x = Cx + Du.
\]

Using symbolic \texttt{matlab}, the transfer function can be calculated as

\[
G(s) = C(sI - A)^{-1}B + D = \frac{2 \Omega m s}{s^2 + \Omega^2},
\]

For the period \( T = 11h = 39600s \), the angular velocity is

\[
\Omega = \frac{2 \pi}{T} = 1.59 \times 10^{-4} \text{rad/s}.
\]

With \( m = 1600kg \), the transfer function \( G(s) \) becomes

\[
G(s) = \frac{2 \times 10^{-7}}{s^2 + 2.5 \times 10^{-8}},
\]

that has no zeros and 2 poles at 0 and \( \pm j1.59 \times 10^{-4} \).

Using \texttt{sisotool} in \texttt{matlab}, a possible controller can be

\[
C(s) = 0.00012 \times \frac{(s + 0.0002)(s + 0.0004)}{(s + 0.004)^2},
\]

which produces the root locus in Figure 7. Note that with the above poles and zeros of the controller, a range of gain \( K \) could be selected such that the closed-loop poles are stable.
Figure 7: Root locus for P6.22.