1 Controllability and Observability

1.1 Linear Time-Invariant (LTI) Systems

State-space:

\[
\dot{x} = Ax + Bu, \quad x(0) = x_0,
\]
\[
y = Cx + Du.
\]

Dimensions:

\[x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p.\]

Notation

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

Transfer function:

\[H(s) = C(sI - A)^{-1}B + D\]

Note that \(H(s)\) is always proper!

Similarity transformation:

\[
\begin{bmatrix}
T^{-1}AT & T^{-1}B \\
CT & D
\end{bmatrix} \sim \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

Similarity does not change transfer function

\[H(s) = CT(sI - T^{-1}AT)^{-1}T^{-1}B + D = C(sI - A)^{-1}B + D\]

System response:

\[Y(s) = \underbrace{H(s)U(s)}_{\text{Input}} + \underbrace{C(sI - A)^{-1}x(0)}_{\text{Initial conditions}}\]

MIMO comes for free!
1.2 Concepts from MAE 280 A

Controllability Matrix:

\[ \mathcal{C}(A, B) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}. \]

Controllability Gramian:

\[ X(t) = \int_0^t e^{A\xi}BB^T e^{A^T\xi}d\xi. \]

Observability Matrix:

\[ \mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \]

Observability Gramian:

\[ Y(t) = \int_0^t e^{A^T\xi}C^TCe^{A\xi}d\xi. \]
1.3 Controllability

Problem: Given $x(0) = 0$ and any $\bar{x}$, can one compute $u(t)$ such that $x(\bar{t}) = \bar{x}$ for some $\bar{t} > 0$?

**Theorem:** The following are equivalent

a) The pair $(A, B)$ is controllable;

b) The Controllability Matrix $C(A, B)$ has full-row rank;

c) There exists no $z \neq 0$ such that $z^*A = \lambda z^*, \quad z^*B = 0$;

d) The Controllability Gramian $X(t)$ is positive definite for some $t \geq 0$.

1.4 Observability

Problem: Given $y(t)$ over $t \in [0, \bar{t}]$ with $\bar{t} > 0$ can one compute $x(t)$ for all $t \in [0, \bar{t}]$?

**Theorem:** The following are equivalent

a) The pair $(A, C)$ is observable;

b) The Observability Matrix $O(A, C)$ has full-column rank;

c) There exists no $x \neq 0$ such that $Ax = \lambda x, \quad Cx = 0$;

d) The Observability Gramian $Y = Y(t)$ is positive definite for some $t \geq 0$.

1.5 Things you should already know

1. Why a) and b) are equivalent.

2. Why can we stop $C(A, B)$ at $A^{n-1}B$ and $O(A, C)$ at $CA^{n-1}$?

3. Kalman canonical forms. E.g. if $(A, C)$ is not observable then

$$
\begin{bmatrix}
A & B \\
C & 0
\end{bmatrix}
\sim
\begin{bmatrix}
A_o & 0 & B_o \\
A_{\bar{o}o} & A_{\bar{o}} & B_{\bar{o}} \\
C_o & 0 & 0
\end{bmatrix}
$$

where $A_o \in \mathbb{R}^{r \times r}$ and $(A_o, C_o)$ is observable.
1.6 The Popov-Belevitch-Hautus Test

**Theorem:** The pair \((A, C)\) is observable if and only if there exists no \(x \neq 0\) such that
\[
Ax = \lambda x, \quad Cx = 0. \tag{1}
\]

**Proof:**

* Sufficiency: Assume there exists \(x \neq 0\) such that (1) holds. Then
\[
CAx = \lambda Cx = 0,
\]
\[
CA^2x = \lambda CAx = 0,
\]
\[
\vdots
\]
\[
CA^{n-1}x = \lambda CA^{n-2}x = 0
\]
so that
\[
\mathcal{O}(A, C)x = 0,
\]
which implies that the pair \((A, C)\) is not observable.

* Necessity: Assume that \((A, C)\) is not observable. Then transform it into the equivalent non observable realization where
\[
\bar{A} = \begin{bmatrix} A_o & 0 \\ A_{\bar{\alpha}} & A_{\bar{\alpha}} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_o & 0 \end{bmatrix}.
\]

Choose \(x \neq 0\) such that
\[
A_{\bar{\alpha}}x = \lambda x.
\]

Then
\[
\begin{bmatrix} A_o & 0 \\ A_{\bar{\alpha}} & A_{\bar{\alpha}} \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ x \end{bmatrix}, \quad \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = 0.
\]
1.7 Controllability Gramian

Problem: Given \(x(0) = 0\) and any \(\bar{x}\), compute \(u(t)\) such that \(x(\bar{t}) = \bar{x}\) for some \(\bar{t} > 0\).

Solution: We know that

\[
\bar{x} = x(\bar{t}) = \int_0^{\bar{t}} e^{A(\bar{t} - \tau)} B u(\tau) d\tau.
\]

If we limit our search to controls \(u\) of the form

\[
u(t) = B^T e^{A^T(t-t)} \bar{z}\]

we have

\[
\bar{x} = \int_0^{\bar{t}} e^{A(\bar{t} - \tau)} BB^T e^{A^T(\bar{t} - \tau)} \bar{z} d\tau,
\]

\[
= \left( \int_0^{\bar{t}} e^{A(\bar{t} - \tau)} B B^T e^{A^T(\bar{t} - \tau)} d\tau \right) \bar{z}, \quad \xi = \bar{t} - \tau
\]

\[
= \left( \int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi \right) \bar{z},
\]

and

\[
\bar{z} = \left( \int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi \right)^{-1} \bar{x},
\]

\[
\Rightarrow u(t) = B^T e^{A^T(\bar{t} - t)} \left( \int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi \right)^{-1} \bar{x}
\]

The symmetric matrix

\[
X(t) := \int_0^t e^{A\xi} B B^T e^{A^T\xi} d\xi
\]

is the Controllability Gramian.
1.8 Stabilizability

Problem: Given any $x(0) = \bar{x}$ can one compute $u(t)$ such that $x(\bar{t}) = 0$ for some $\bar{t} > 0$?

Theorem: The following are equivalent

a) The pair $(A, B)$ is stabilizable;

b) There exists no $z \neq 0$ and $\lambda$ such that $z^*A = \lambda z^*$, $z^*B = 0$ with $\lambda + \lambda^* \geq 0$.

1.9 Detectability

Problem: Given $y(t)$ over $t \in [0, \bar{t}]$ with $\bar{t} > 0$ can one compute $x(\bar{t})$?

Theorem: The following are equivalent

a) The pair $(A, C)$ is detectable;

b) There exists no $x \neq 0$ and $\lambda$ such that $Ax = \lambda x$, $Cx = 0$ with $\lambda + \lambda^* \geq 0$. 
1.10 Example: satellite in circular orbit

Satellite of mass $m$ with thrust in the radial direction $u_1$ and in the tangential direction $u_2$. From Skelton, DSC, p. 101.

Newton’s law

$$m \ddot{r} = \vec{u}_1 + \vec{u}_2 + \vec{f}_g,$$

where $\vec{f}_g$ is the gravitational force

$$\vec{f}_g = -\frac{km}{r^2} \vec{r}.$$

Using cylindrical coordinates

$$\vec{e}_1 = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix},$$

we have

$$\vec{r} = r \vec{e}_1, \quad \vec{u}_1 = u_1 \vec{e}_1, \quad \vec{u}_2 = u_2 \vec{e}_2, \quad \vec{f}_g = -\frac{km}{r^2} \vec{e}_1.$$
We need to compute
\[
\ddot{\mathbf{r}} = \frac{d^2}{dt^2}(\ddot{r}\mathbf{e}_1) = \frac{d}{dt}(\dot{r}\dot{\mathbf{e}}_1 + r\ddot{\mathbf{e}}_1) = \ddot{r}\mathbf{e}_1 + 2\dot{r}\dot{\mathbf{e}}_1 + r\dddot{\mathbf{e}}_1,
\]
where
\[
\dot{\mathbf{e}}_1 = \dot{\theta} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} = \dot{\theta}\mathbf{e}_2,
\]
\[
\dddot{\mathbf{e}}_1 = \ddot{\theta} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} + \dot{\theta}^2 \begin{pmatrix} -\cos \theta \\ -\sin \theta \end{pmatrix} = \ddot{\theta}\mathbf{e}_2 - \dot{\theta}^2\mathbf{e}_1.
\]
That is
\[
\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_1 + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_2,
\]
so that Newton’s law can be rewritten as
\[
m(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_1 + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_2 = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 - \frac{km}{r^2}\mathbf{e}_1,
\]
or, equivalently, as the two scalar differential equations
\[
m(\ddot{r} - r\dot{\theta}^2) = u_1 - \frac{km}{r^2},
\]
\[
m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = u_2.
\]
In state space
\[
x = \begin{pmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{pmatrix}, \quad \dot{x} = \begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \ddot{r} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_1x_4^2 - k/x_1^2 \\ -2x_3x_4/x_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(mx_1) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]
This is a nonlinear system and we look for equilibrium ($\ddot{r} = \ddot{\theta} = 0$) when $u_1 = u_2 = 0$. This can be stated as
\[
x_1x_4^2 - k/x_1^2 = 0, \quad -2x_3x_4/x_1 = 0.
\]
The second condition implies $x_3 = \dot{r}$ and/or $x_4 = \dot{\theta}$ must be zero. We choose $x_3 = \dot{r} = 0$ which implies $x_1 = r = \bar{r}$ constant and
\[
x_4 = \dot{\theta} = \sqrt{\frac{k}{x_3^2}} = \sqrt{\frac{k}{\bar{r}^3}} = \bar{\omega} \quad \Rightarrow \quad k = \bar{r}^3\bar{\omega}^2.
\]
Note also that $x_2 = \theta = \bar{\omega}t$. 

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This nonlinear system is in the form
\[ \dot{x} = f(x, t) + g(x)u. \]
We will linearize \( f(x, t) \) and \( g(x)u \) around the equilibrium point \((\bar{x}, \bar{u})\) to obtain the linearized system
\[ \dot{x} = (\nabla f_x)^T [x(t) - \bar{x}(t)] + (\nabla g_x)^T [x(t) - \bar{x}(t)]\bar{u} + g(\bar{x})u. \]
For this problem
\[
\bar{x}(t) = \begin{pmatrix} \bar{r} \\ \bar{\omega}t \\ 0 \\ \bar{\omega} \end{pmatrix}, \quad \bar{u} = 0, \quad f(x, t) = \begin{pmatrix} x_3 \\ x_4 \\ x_1x_4^2 - k/x_1^2 \\ -2x_3x_4/x_1 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(mx_1) \end{pmatrix},
\]
and
\[
(\nabla f_x)^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2k/r^3 + \bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \end{bmatrix}.
\]
This produces the linearized system
\[
\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(mr) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]
If we looking at the satellite (from the earth) we can say that we can observe \( r \) and \( \dot{\theta} \) (how?), that is
\[
y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x.
\]
Questions:

1) Can we estimate the state of the satellite by measuring only \( r \)?
2) Can we estimate the state of the satellite by measuring only \( \dot{\theta} \)?
3) Can we estimate the state of the satellite by measuring \( r \) and \( \dot{\theta} \)?
4) Can the system be controlled to remain in circular orbit using radial thrusting \( (u_1) \) alone?
5) Can the system be controlled using tangential thrusting \( (u_2) \) alone?
Question: Can we estimate the state of the satellite by measuring only $r$?

Answer: Is the system observable when $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$? Compute the observability matrix

$$O(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix},$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -\bar{\omega}^2 & 0 \end{bmatrix}$$

Physical interpretation: measuring $r$ does not give any information on $\theta$ or $\bar{\theta}$!

Note that if we know the satellite is in equilibrium and “measure” $k$ then

$$\dot{\theta} = \bar{\omega} = \sqrt{\frac{k}{\bar{r}^3}}.$$

But we still do not know $\theta$ since

$$\theta(t) = \theta(0) + \omega t,$$

and we do not know $\theta(0)$!
Question: Can we estimate the state of the satellite by observing $\dot{\theta}$ only?

Answer: Is the system observable when $C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$? Compute the observability matrix

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix},$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \\ -6\bar{\omega}^3/\bar{r} & 0 & 0 & -4\bar{\omega}^2 \\ 0 & 0 & 2\bar{\omega}^3/\bar{r} & 0 \end{bmatrix}$$

Physical interpretation: again, if we try to reconstruct $\theta$ from $\dot{\theta}$ we still need to know $\theta$ at some $\bar{t}$! From that point on

$$\theta = \theta(\bar{t}) + \int_{\bar{t}}^{t} \dot{\theta}(\tau) d\tau.$$
Question: Can we estimate the state of the satellite by measuring $r$ and $\dot{\theta}$?

Answer: Is the system observable when $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$? Compute the observability matrix

$$O(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix},$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & -2\bar{\omega}/\bar{r} & 0 \\
3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\
-6\bar{\omega}^3/\bar{r} & 0 & 0 & -4\omega^2 \\
0 & 0 & -\bar{\omega}^2 & 0 \\
0 & 0 & 2\bar{\omega}^3/\bar{r} & 0
\end{bmatrix}.$$

Physical interpretation: can we estimate $\theta$ at all?
Question: Can the system be controlled to remain in circular orbit using radial thrusting \((u_1)\) alone?

Answer: Is the system controllable when \(B = \begin{bmatrix} 0 \\ 0 \\ 1/m \\ 0 \end{bmatrix}\)? Compute the controllability matrix

\[
C(A, B) = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix},
\]

\[
= \frac{1}{m} \begin{bmatrix}
 0 & 1 & 0 & -\bar{\omega}^2 \\
 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \\
 1 & 0 & -\bar{\omega}^2 & 0 \\
 0 & -2\bar{\omega}/\bar{r} & 0 & 2\bar{\omega}^3/\bar{r}
\end{bmatrix}
\]

Note that

\[
-\bar{\omega}^2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2\bar{\omega}/\bar{r} \end{pmatrix} = \begin{pmatrix} -\bar{\omega}^2 \\ 0 \\ 0 \\ 2\bar{\omega}^3/\bar{r} \end{pmatrix}
\]

which implies that the system is not controllable from \(u_1\)!

Physical interpretation: there must be a change in the angular velocity \(\dot{\theta}\) if one changes the radius!
Question: Can the system be controlled using tangential thrusting \((u_2)\) alone?

Answer: Is the system controllable when
\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
1/(m\bar{r})
\end{bmatrix}
\]

? Compute the controllability matrix
\[
C(A, B) = \begin{bmatrix}
B & AB & A^2B & A^3B
\end{bmatrix},
\]

\[
= \frac{1}{m\bar{r}} \begin{bmatrix}
0 & 0 & 2\bar{r}\bar{\omega} & 0 \\
0 & 1 & 0 & -4\bar{\omega}^2 \\
0 & 2\bar{r}\bar{\omega} & 0 & -2\bar{r}\bar{\omega}^3 \\
1 & 0 & -4\bar{\omega}^2 & 0
\end{bmatrix}
\]

The above matrix has full rank, so the system is controllable from \(u_2\)!

Physical interpretation: we can change the radius by changing the angular velocity!
1.11 More on Gramians

**Theorem:** The Controllability Gramian

\[ X(t) = \int_0^t e^{A\xi} BB^T e^{AT\xi} d\xi, \]

is the solution to the differential equation

\[ \frac{d}{dt} X(t) = AX(t) + X(t)AT + BB^T. \]

If \( X = \lim_{t \to \infty} X(t) \) exists then

\[ AX + XAT + BB^T = 0. \]

**Theorem:** The Observability Gramian

\[ Y(t) = \int_0^t e^{AT\xi} C^T Ce^{A\xi} d\xi, \]

is the solution to the differential equation

\[ \frac{d}{dt} Y(t) = A^T Y(t) + Y(t)A + C^T C. \]

If \( Y = \lim_{t \to \infty} X(t) \) exists then

\[ A^T Y + YA + C^T C = 0. \]
Proof (Controllability): For the first part, compute
\[
\frac{d}{dt} X(t) = \frac{d}{dt} \int_0^t e^{A\xi} B B^T e^{A^T\xi} d\xi = \frac{d}{dt} \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau,
\]
\[
= \int_0^t \frac{d}{dt} e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} + e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} \bigg|_{\tau=t},
\]
\[
= A \left( \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau \right)
\]
\[
+ \left( \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau \right) A^T + B B^T,
\]
\[
= A X(t) + X(t) A^T + B B^T.
\]

For the second part, use the fact that \(X(t)\) is smooth and therefore
\[
\lim_{t \to \infty} X(t) = X \implies \lim_{t \to \infty} \frac{d}{dt} X(t) = 0.
\]
**Lemma:** Consider the Lyapunov Equation

\[ A^T X + XA + C^T C = 0 \]

where \( A \in \mathbb{C}^{n \times n} \) and \( C \in \mathbb{C}^{m \times n} \).

1. A solution \( X \in \mathbb{C}^{n \times n} \) exists and is unique if and only if \( \lambda_j(A) + \lambda_i^*(A) \neq 0 \) for all \( i, j = 1, \ldots, n \). Furthermore \( X \) is symmetric.

2. If \( A \) is Hurwitz then \( X \) is positive semidefinite.

3. If \((A, C)\) is detectable and \( X \) is positive semidefinite then \( A \) is Hurwitz.

4. If \((A, C)\) is observable and \( A \) is Hurwitz then \( X \) is positive definite.

**Proof:**

**Item 1.** The Lyapunov Equation is a linear equation and it has a unique solution if and only if the homogeneous equation associated with the Lyapunov equation admits only the trivial solution. Assume it does not, that is, there \( \bar{X} \neq 0 \) such that

\[ A^T \bar{X} + \bar{X} A = 0 \]

Then, multiplication of the above on the right by \( x_i^* \neq 0 \), the \( i \)-th eigenvector of \( A \) and on the right by \( x_j^* \neq 0 \) yields

\[ 0 = x_i^* A^T \bar{X} x_j + x_j^* \bar{X} A x_j = [\lambda_j(A) + \lambda_i^*(A)] x_i^* \bar{X} x_j. \]

Since \( \lambda_i(A) + \lambda_j(B) \neq 0 \) by hypothesis we must have \( x_i^* \bar{X} x_j = 0 \) for all \( i, j \). One can show that this indeed implies \( \bar{X} = 0 \), establishing a contradiction.

That \( X \) is symmetric follows from uniqueness since

\[ 0 = (A^T X + XA + C^T C)^T - (A^T X + XA + C^T C) = A^T (X^T - X) + (X^T - X) A \]

so that \( X^T - X = 0 \).

**Item 2.** If \( A \) is Hurwitz then \( \lim_{t \to \infty} e^{At} = 0 \). But

\[ X = \int_0^\infty e^{A^T t} C^T C e^{At} dt \geq 0 \]

and

\[ A^T X + XA = \lim_{t \to \infty} \int_0^\infty \frac{d}{dt} e^{A^T t} C^T C e^{At} dt = e^{A^T t} C^T C e^{At} \bigg|_0^\infty = -C^T C. \]
Item 3. Assume \((A, C)\) is detectable, \(X \succeq 0\) and that \(A\) is not Hurwitz. Then there exists \(\lambda\) and \(x \neq 0\) such that \(Ax = \lambda x\) with \(\lambda + \lambda^* \geq 0\). But if \(X\) solves the Lyapunov equation

\[-x^*C^T Cx = x^* A^T X x + x^* X A x = (\lambda + \lambda^*) x^* X x \geq 0\]

which implies \(C x = 0\), hence \((A, C)\) not detectable.

Item 4. Assume that \((A, C)\) is observable and \(A\) is Hurwitz. From Item 2. \(X \succeq 0\). Assume \(X\) is not positive definite, that is, there exists \(\bar{x} \neq 0\) such that \(X \bar{x} = 0\). It follows that

\[0 = \bar{x}^* X \bar{x} = \int_0^\infty \bar{x}^* e^{A^T t} C^T C e^{A t} \bar{x} \, dt = \int_0^\infty y^*(t) y(t) \, dt\]

which implies that response \(y(t) = C e^{A t} \bar{x} = 0\) to a non null initial condition \(x(0) = \bar{x}\) is null, which contradicts the hypothesis that \((A, C)\) is observable.